On the shape of the ground state eigenvalue density of a random Hill's equation

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Abstract

Consider the Hill's operator $Q = -d^2/dx^2 + q(x)$ in which $q(x)$, $0 \le x \le 1$, is a White Noise. Denote by $f(\mu)$ the probability density function of $-\lambda_0(q)$, the negative of the ground state eigenvalue, at μ. We prove the detailed asymptotics:

$$f(\mu) = \frac{4}{3\pi}\,\mu\,\exp\left[-\frac{8}{3}\mu^{3/2} - \frac{1}{2}\mu^{1/2}\right](1 + o(1))$$

as $\mu \to +\infty$. This result is based on a precise Laplace analysis of a functional integral representation for $f(\mu)$ established by S. Cambronero and H.P. McKean in [5].

1 Introduction

We consider fluctuations of the ground state eigenvalue of a random Hill's operator $Q = -d^2/dx^2 + q(x)$ in which the periodicity is fixed at one and the potential $q(x)$ is a White Noise. Formally, $q(x) = b'(x)$ for a standard Brownian Motion $b(x)$, and the eigenvalue problem $Q\psi = \lambda\psi$ must be interpreted in the sense of Itô. In those terms it reads $d\psi'(x) = db(x) - \lambda\psi(x)dx$, and there is no problem solving for ψ in the space $C^{3/2-}$.

Let $\lambda_0(q)$ denote the ground state eigenvalue of Q. Our jumping off point is an explicit formula for the law of this object due to the authors of [5]. Choosing the sign for later convenience we further denote by $f(\mu)$ the probability density function of $-\lambda_0(q)$ at the point μ. Then, the result of [5] is

$$f(\mu) = \frac{1}{\sqrt{2\pi}} \int_H \exp\left\{-\frac{1}{2}\int_0^1 (\mu - p^2(x))^2 dx\right\} A(p)\, dP_0 \tag{1.1}$$

where

$$A(p) = \int_0^1 e^{2\int_0^x p(x')dx'} dx \times \int_0^1 e^{-2\int_0^x p(x')dx'} dx, \tag{1.2}$$

and P_0 is a probability measure on H, the space of continuous functions of period one and mean zero. More specifically, P_0 is the Circular Brownian Motion (CBM) $p(x)$, $0 \le x \le 1$, conditioned so that $\int_0^1 p(x)dx = 0$. CBM, we recall, is the measure on periodic paths formed from the standard Brownian Motion starting from $p(0) = c$, conditioned to return to c at $x = 1$, with this common starting/ending point distributed over the line according to Lebesgue measure. In other words, for any event $\mathcal{A}$ of the path, $CBM(\mathcal{A}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} BM_{00}(\mathcal{A} + c)dc$ in which BM_{00} denotes the mean of the Brownian Bridge

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of length one. While CBM itself has infinite total mass, P_0 is the distribution of an honest rotation invariant Gaussian process: a short computation will show that

$$E_0\Big[F(p)\Big] = BM_{00}\Big[F(p - \int_0^1 p)\Big]$$

for any bounded measurable test function $F = F(p(x) : 0 \le x \le 1)$.

The functional integral representation of the density $f(\mu)$ given by (1.1) is based on a correspondence between Hill's equation and Ricatti's equation. To explain, first fix a λ to the left of $\lambda_0(q)$. Then it is well known (see [18]) that Hill's equation possesses a positive solution ψ with multiplier $m > 0$: $-\psi'' + q\psi = \lambda\psi$ and $\psi(x+1) = m\psi(x)$. Further, the logarithmic derivative $p = \psi'/\psi$ is a periodic solution of Ricatti's equation $q = \lambda + p' + p^2$ with $\int_0^1 p \ge 0$. Now view this *Ricatti correspondence* as a map between measure spaces. Under this map the restriction of the White Noise measure to $\{\lambda \le \lambda_0(q)\}$ is identified with the restriction of the CBM to $\{\int_0^1 p \ge 0\}$ up to a suitable Jacobian factor. The computation of that Jacobian is the chief accomplishment of [5], resulting in, among other things, the formula (1.1).

Given (1.1), our purpose here is to describe the shape of the density. That is, we investigate the detailed asymptotics of $f(\mu)$ as $\mu \to \pm\infty$. The left tail has in fact already been discussed in [5]: they remark that

$$f(\mu) = \sqrt{\frac{(-\mu)}{\pi}} \exp\Big[-\frac{1}{2}\mu^2 - \frac{1}{\sqrt{2}}(-\mu)^{1/2}\Big](1 + o(1)) \quad \text{for } \mu \to -\infty.$$

This is easy to understand. The exponent is expanded out as in $\int_0^1 (|\mu| + p^2)^2 = \mu^2 - 2|\mu| \int_0^1 p^2 - \int_0^1 p^4$. Next, $A(p)$ and $\exp\{-(1/2)\int_0^1 p^4\}$ are shown to be negligible when compared to the coercive Gaussian weight $\exp\{-|\mu| \int_0^1 p^2\}$. The derivation is completed by computing $E_0[\exp\{-|\mu| \int_0^1 p^2\}]$ exactly.

The analysis of right tail is far more involved. Our result is the following.

Theorem 1.1 *The probability density of $-\lambda_0(q)$ has the shape*

$$f(\mu) = \frac{4}{3\pi}\,\mu\, \exp\Big[-\frac{8}{3}\mu^{3/2} - \frac{1}{2}\mu^{1/2}\Big](1 + o(1)) \quad \text{for } \mu \to +\infty. \tag{1.3}$$

While the limit $\mu \to -\infty$ concentrates the path at the unique and trivial trajectory $p \equiv 0$, taking $\mu \to +\infty$ leads to a completely different picture. In this regime it is most advantageous for p to live near $\pm\sqrt{\mu}$. However, because of the restriction to $\int_0^1 p = 0$, the path is forced to divide its favors more or less evenly between the two choices of sign. Furthermore, due to the rotation invariance of CBM, any translation of an extremal path is also extremal. Thus, for $\mu \to +\infty$, we are dealing with Laplace asymptotics of a degenerate function space integral. Even so, we are able to obtain beyond leading order information.

There is of course no shortage of investigations into the precise large deviations of Wiener-type or other functional integrals, including cases in with the underlying functional possesses degeneracies. Important examples are [6], [3], [2], [14] and [15]. Nevertheless, the present problem has various features which set it apart and require the analysis to be done "by hand". First, the large parameter enters $f(\mu)$ in a fundamentally different way than is assumed throughout the cited list. More importantly, those dealing with degenerate problems assume nondegeneracy in directions orthogonal to the extremal set. Our integral does not posses this property; there exists a more subtle degeneracy besides that stemming from the translation invariance.

Random Schrödinger operators of the type Q arise in models of disordered solids as is explained in the comprehensive book [17]. The White Noise potential offers a simplifying caricature. Its use goes back to [7], but see also [9] and [8] which discuss the integrated density of states. A description of the ground state energy is of separate importance. In the present White Noise setting with dimension equal to one, [20] proves a limit law for $-\lambda_0(q)$ as the periodicity is taken to infinity. Also in the

thermodynamic regime, the study of the almost sure behavior of the ground state in any dimension subject to a potential of Poisson or Gibbs type is well developed: see [24] and [19] and references therein. Still, the understanding of the actual distribution in a finite volume with any kind of potential remains in its infancy, and the result described above prompts further inquiry. In particular, it is reasonable to ask to what extent the shape of $f(\mu)$ is universal for some class of rough potentials. The Gaussian/sub-Gaussian tails to the left/right seen here have an intuitive explanation: level repulsion holding down the left tail, with the ground state free to take advantage of deep wells created by the White Noise to the right.

The rest of the paper takes the following course. In Section 2 we study the associated rate function and discuss a leading order result of the form $\mu^{-3/2}\log f(\mu) \simeq -8/3$ for $\mu \uparrow \infty$. Asymptotics at this level are accounted for by a vicinity of a one parameter family of paths (the degeneracy). Section 3 outlines how to expand about the set of extrema. Namely, the degeneracy is dealt with by a conditioning procedure, leaving an integral with respect to a certain Gaussian measure as the principle lower order term. Using a connection to a particular (deterministic) Hill operator, required properties of this Gaussian measure are collected in Section 4. The calculation is picked up again in Section 5 which contains the main error estimate: here we dispose of the terms beyond the Gaussian correction. Afterward, in Section 6, the Gaussian correction is computed exactly. In essence this completes our calculation. Section 7 gathers the results through that point and states Theorem 7.1 which, in some sense, is a more accurate statement of the main result. It is then explained how asymptotics of the various quantities appearing in Theorem 7.1 translate that result into the above Theorem 1.1. Finally, Section 8 serves as an appendix containing various technicalities needed along the way.

2 Leading order asymptotics

2.1 The rate function

At an exponential scale, the asymptotics of integrals of the type (1.1) are well understood to be associated with a characteristic variational problem. In the case of $f(\mu)$, that problem is to minimize

$$I_\mu(p) \equiv \frac{1}{2}\int_0^1 \Big(\mu - p^2(x)\Big)^2 dx + \frac{1}{2}\int_0^1 (p'(x))^2 dx \tag{2.1}$$

over periodic functions of mean zero. Scaling as in $p(\cdot)\to\frac{1}{\sqrt{\mu}}f(\cdot/\sqrt{\mu})$ we find that

$$\begin{aligned} \inf_{p\in H} I_\mu(p) &= \mu^{3/2}\,\inf_{f\in H_a}\Big\{\frac{1}{2}\int_{-a}^{a}(1-f^2(x))^2dx + \frac{1}{2}\int_{-a}^{a}(f'(x))^2dx\Big\} \\ &\equiv \mu^{3/2}\,\inf_{f\in H_a} I(f;a) \end{aligned} \tag{2.2}$$

in which $a = \sqrt{\mu}/2$ and now H_a is the class of periodic C^1 functions satisfying $\int_{-a}^{a} f = 0$. This already suggests the 3/2-power in the exponent of (1.3). As to its 8/3-multiplier and further properties of the *rate function* I we have the following.

Theorem 2.1 *The infimum $I^*(a) = \inf_{f\in H_a} I(f;a)$ is attained. Further, $I^*(a) \le 8/3$ for all $a > 0$ and $I^*(a)\to 8/3$ as $a\to\infty$.*

Proof As the level sets $\{I(\cdot;a) \le K\}$ are weakly compact in H^1 and strongly so in L^∞, the existence of a minimizer poses no problem. For the upper bound on $I^*(a)$, consider the test function

$$f_a(x) = \begin{cases} -\tanh(x+a) & \text{for} \quad -a \le x < -a/2, \\ \tanh(x) & \text{for} \quad -a/2 \le x < a/2, \\ -\tanh(x-a) & \text{for} \quad a/2 \le x < a. \end{cases} \tag{2.3}$$

Then,

$$\begin{aligned} I^*(a) \le I(f_a; a) &= \int_{-a/2}^{a/2} (1 - f_a^2(x))^2 dx + \int_{-a/2}^{a/2} (f_a'(x))^2 \\ &\le \int_{-\infty}^{\infty} (1 - \tanh^2(x))^2 dx + \int_{-\infty}^{\infty} (\tanh'(x))^2 dx \\ &= 2 \int_{-\infty}^{\infty} \operatorname{sech}^4(x) dx = 8/3. \end{aligned}$$

As for the convergence $I^*(a) \to 8/3$, computing a first variation of $I(\cdot; a)$ shows that any minimizer satisfies $f'' = 2f^3 - 2f^2 - \alpha$ where α is the Lagrange multiplier corresponding to the condition $\int_{-a}^{a} f = 0$. Multiplying by f' and integrating, this may be brought into the form

$$\frac{1}{2}(f')^2 = \frac{1}{2} f^4 - f^2 - \alpha f + \frac{1}{2}\beta \tag{2.4}$$

with a new constant β. We will show in the appendix that, as $a \to \infty$, $\alpha \to 0$ and $\beta \to 1$.

Now choose a minimizer f_a^* for each $a \gg 1$. Since every f_a^* must have at least two roots on account of being mean zero, by rotation invariance we can assume that $f_a^*(0) = 0$ for all $\{f_a^*\}$. Next, for any large fixed $M > 0$ less than $a/2$ we certainly have that

$$I^*(a) = \frac{1}{2} \int_{-a}^{a} (df_a^*/dx)^2 + \frac{1}{2} \int_{-a}^{a} (1 - (f_a^*)^2)^2 \ge \frac{1}{2} \int_{-M}^{M} (df_a^*/dx)^2 + \frac{1}{2} \int_{-M}^{M} (1 - (f_a^*)^2)^2.$$

It follows that the sequence $\{f_a^*\}$ is bounded in $H^1 \cap L^\infty[-M, M]$ and so has a subsequence converging weakly in H^1 and strongly in L^4. By writing the relation (2.4) in the form

$$f' = \sqrt{f^4 - 2f^2 - 2\alpha f + \beta}, \tag{2.5}$$

using its weak formulation and the regularity theory that comes with it, we find that the above convergence is sufficient to conclude that any limit satisfies $f_\infty' = 1 - f_\infty^2$ over $[-M, M]$ and $f_\infty(0) = 0$. That is, $f_\infty = \tanh$.

Since the centered $\{f_a^*\}$ converge to f_∞ on an interval $[-M, M]$ for any choice of $M > 0$, it follows that the distance between any two zeros of f_a^* must be tending to infinity as $a \to \infty$. Therefore, we may also isolate a symmetric interval of length $2M$ about a second zero of f_a^*. On this second interval we will have the same type of convergence (by precisely the same arguments). Then, by adding both contributions we also conclude that

$$I^*(a) \ge \int_{-M}^{M} (df_a^*/dx)^2 + \int_{-M}^{M} (1 - (f_a^*)^2)^2$$

for all a large enough. This results in

$$\liminf_{a \to \infty} I^*(a) \ge 2 \int_{-M}^{M} \operatorname{sech}^4(x) dx \uparrow 8/3$$

upon letting $M \uparrow \infty$ afterward. The proof is finished.

2.2 Large Deviations

While the previous result serves as a guide, the next step is to extract the $\exp[-8/3\mu^{3/2}]$ behavior from the integral $f(\mu)$.

Theorem 2.2 *We have the leading order asymptotics:*

$$\limsup_{\mu\to\infty} \mu^{-3/2} \log \int_H e^{-\frac{1}{2}\int_0^1 (\mu - p^2)^2} A(p)\ dP_0 \le -8/3.$$

More important in the sequel, we show that one has sharper decay of the same order when the integral is restricted to a set away from any I_μ minimizer. Define

$$C_\varepsilon^\mu = \Big\{ p \in H : d(p, \mathcal{M}) \le \varepsilon \sqrt{\mu} \Big\} \tag{2.6}$$

in which $\mathcal{M}$ is the set of minimizers of I_μ and $d(p, \mathcal{A})$ is the distance between a path p and a set $\mathcal{A}$ in sup-norm. Then we have:

Theorem 2.3 *There exists a $\eta > 0$ depending on ε and a constant C so that*

$$\int_{H \setminus C_\varepsilon^\mu} e^{-\frac{1}{2}\int_0^1 (\mu - p^2)^2} A(p)\ dP_0 \le C e^{-(8/3+\eta)\mu^{3/2}}$$

for all μ large enough.

Proof of Theorem 2.2 Matters are simplified by noticing that it is enough to prove that

$$\limsup_{\mu\to\infty} \mu^{-3/2} \log \int_H e^{-\frac{1}{2}\int_0^1 \left(\mu - p^2\right)^2} dP_0 \le -\frac{8}{3}. \tag{2.7}$$

This is because $A(p) \le \exp[2\sqrt{\int_0^1 p^2}] \le \exp[1 + \int_0^1 p^2]$ which implies the upper bound

$$\int_H e^{-\frac{1}{2}\int_0^1 \left(\mu - p^2\right)^2} A\left(p\right) dP_0 \le e^{\mu+3/2} \int_H e^{-\frac{1}{2}\int_0^1 \left(\mu + 1 - p^2\right)^2} dP_0,$$

the prefactor $e^{\mu+3/2}$ being irrelevant in the present scale.

The proof of (2.7) is split into several steps. First we define the set of paths

$$H^\mu(\gamma, \eta) = \left\{ p \in H : \left| \frac{1}{2} \int_0^1 \left(\mu - p^2\right)^2 - \gamma \mu^{3/2} \right| < \eta \mu^{3/2} \right\} \tag{2.8}$$

for any positive η and γ. As we shall see, restricted to such a set, the P_0 integral of $\exp\left[-(1/2)\int_0^1 (\mu - p^2)^2\right]$ is easy to control through the variational problem studied Section 2.1. For this reason we make the decomposition

$$\int_H e^{-\frac{1}{2}\int_0^1 \left(\mu - p^2\right)^2} dP_0 \le \sum_{0 \le k \le 8/3\eta} \int_{H^\mu(\eta k, \eta)} e^{-\frac{1}{2}\int_0^1 \left(\mu - p^2\right)^2} dP_0 + \exp\left(-\frac{8}{3}\mu^{3/2}\right) \tag{2.9}$$

after which we may invoke the bound

$$\int_{H^\mu(\gamma,\eta)} e^{-\frac{1}{2}\int_0^1 \left(\mu - p^2\right)^2} dP_0 \le \exp\left[\left(-\gamma + \eta\right)\mu^{3/2}\right] P_0\Big(H^\mu(\gamma, \eta)\Big) \tag{2.10}$$

in each integral on the right hand side of (2.9). This follows directly from the definition of $H^\mu(\gamma, \eta)$.

The upshot is that we must now control the probabilities $P_0(H^\mu(\gamma, \eta))$. Toward this end we overestimate further as in

$$\begin{aligned} P_0\Big(H^\mu(\gamma, \eta)\Big) \quad &\le \quad P_0\Big(\frac{1}{2}\int_0^1 \left(\mu - p^2\right)^2 \le \left(\gamma + \eta\right)\mu^{3/2}\Big) \\ &\equiv \quad P_0^\mu\Big(\frac{1}{2}\int_0^1 \left(1 - p^2\right)^2 \le \left(\gamma + \eta\right)\frac{1}{\sqrt{\mu}}\Big) \end{aligned} \tag{2.11}$$

where we have introduced the scaled measure $P_0^\mu(p \in A) \equiv P_0(\frac{1}{\sqrt{\mu}} p \in A)$. Under this scaling, the previous display reads $P_0(H^\mu(\gamma,\eta)) \le P_0^\mu(D^\mu(\gamma,\eta))$, and

$$D^\mu = D^\mu(\gamma,\eta) \equiv \left\{ p \in H : \frac{1}{2}\int_0^1 (1-p^2)^2 \le (\gamma+\eta)\,\frac{1}{\sqrt{\mu}} \right\} \tag{2.12}$$

marks yet another definition.

The next step, estimating the P_0^μ probability of D^μ, is accomplished by discretizing the path. This is a common procedure, see for example the proof of Schilder's Theorem in [25]. Let p_n be the polygonal path determined by the values $p(k/n)$ at k/n for $k = 0,\dots,n$ and introduce $\widehat{p}_n = p_n - \int_0^1 p_n$ to force things to reside in H. Now, for whatever set $D \subseteq H$ we have that, if $p \in D$, then the polygonal p_n is either very close to p or far away from D. In symbols

$$P_0^\mu(D) \le P_0^\mu\Big(||p - \widehat{p}_n||_\infty \ge \delta \Big) + P_0^\mu\Big(\widehat{p}_n \in D_\delta \Big) \tag{2.13}$$

in which $||p||_\infty = \sup_{0 \le x < 1} |p(x)|$ for any path $p \in H$ and D_δ is the δ-enlargement of D in that norm. That is, $q \in D_\delta$ when $\inf_{p \in D} ||q - p||_\infty \le \delta$. We next tackle each term of the right of (2.13).

For the deviation between p and its approximate $\widehat{p}_n$ we first note

$$P_0^\mu\Big(\|p - \widehat{p}_n\|_\infty \ge \delta \Big) \le P_0^\mu\Big(\|p - p_n\|_\infty \ge \frac{\delta}{2} \Big),$$

because $\|p - \widehat{p}_n\|_\infty \le \|p - p_n\|_\infty + |\int_0^1 (p - p_n)| \le 2\,\|p - p_n\|_\infty$. Then,

$$\begin{aligned} P_0^\mu\Big(\|p - p_n\|_\infty \ge \delta/2 \Big) &\le P_0^\mu\Big(\bigcup_{k=0}^{n-1} \Big\{ \sup_{\frac{k}{n} \le x \le \frac{k+1}{n}} |p(x) - p(k/n)| \ge \delta/4 \Big\} \Big) \\ &\le n P_0^\mu\Big(\sup_{0 \le x \le 1/n} |p(x) - p(0)| \ge \delta/4 \Big), \end{aligned} \tag{2.14}$$

having used the rotation invariance of CBM in line two. Next we recall the definition of the measure P_0^μ and write

$$\begin{aligned} P_0^\mu\Big(\|p - p_n\|_\infty \ge \delta/2 \Big) &\le nBM_{00}\Big(\sup_{0 \le x \le 1/n} |p(x)| \ge \sqrt{\mu}\delta/4 \Big) \\ &\le 2nBM_0\Big(\sup_{0 \le x \le 1/n} |p(x)| \ge \sqrt{\mu}\delta/4 \Big) \le 32(\sqrt{n}/\delta\sqrt{\mu}) e^{-n\mu\delta^2/32}. \end{aligned} \tag{2.15}$$

The first and third inequalities require no explanation. For the middle inequality, note that if $\mathcal{A}$ is an event measurable over $\{p(x), 0 \le x \le 3/4\}$, then $BM_{00}(\mathcal{A}) = 2BM_0[e^{-p^2(3/4)/2}, \mathcal{A}] \le 2P_0(\mathcal{A})$.

For the second term in (2.13), bring in $\mathcal{I}(w) = \frac{1}{2}\int_0^1 |w'|^2$ the usual Brownian rate function ($\mathcal{I}(w) \equiv \infty$ when the integrand is not defined) and $\mathcal{I}(D) = \inf_{w \in D} \mathcal{I}(w)$. Again, we first move to the Bridge measure:

$$\begin{aligned} P_0^\mu\Big(\widehat{p}_n \in D \Big) &\le P_0^\mu\Big(\mathcal{I}(\widehat{p}_n) \ge \mathcal{I}(D) \Big) \\ &= BM_{00}\Big(\frac{n}{2} \sum_{k=0}^{n-1} |p((k+1)/n) - p(k/n)|^2 \ge \mu \mathcal{I}(D) \Big). \end{aligned}$$

The latter probability may be written out explicitly, and we continue to overestimate: with $S_n(x) = (n/2)(x_1^2 + \sum_{k=1}^{n-2} (x_{k+1} - x_k)^2 + x_{n-1}^2)$,

$$BM_{00}\Big(\frac{n}{2} \sum_{k=0}^{n-1} |p(k+1/n) - p(k/n)|^2 \ge \mu \mathcal{I}(D) \Big) \tag{2.16}$$

$$
\begin{aligned}
&= \frac{1}{(2\pi/n)^{(n-1)/2}} \int_{S_n(x) \geq \mu \mathcal{I}(D)} \exp\Big[-S_n(x)\Big] dx_1 \cdots dx_{n-1} \\
&\leq e^{-(1-\eta)\mu \mathcal{I}(D)} \frac{1}{(2\pi/n)^{(n-1)/2}} \int_{\mathbb{R}^{n-1}} \exp\Big[-\eta S_n(x)\Big] dx_1 \cdots dx_{n-1} \\
&= \Big(\frac{1}{\sqrt{\eta}}\Big)^{n-1} e^{-(1-\eta)\mu \mathcal{I}(D)}.
\end{aligned}
$$

Returning now to the event of interest (2.12), we combine the two bounds (2.15) and (2.16) to find that

$$
P_0^\mu\Big(D_\delta^\mu(\gamma,\eta)\Big) \leq 32 \frac{\sqrt{n}}{\delta\sqrt{\mu}} \exp\Big[-n\mu\delta^2/32\Big] + (\sqrt{\eta})^{1-n} \exp\Big[-(1-\eta)\mu \mathcal{I}\Big(D_\delta^\mu(\gamma,\eta)\Big)\Big]. \tag{2.17}
$$

The first term on the right can be made small by choosing n appropriately. It remains to estimate $\mathcal{I}(D_\delta^\mu)$ from below. The results of the Section 2.1 imply that, in the present scaling,

$$
\frac{1}{2\mu} \int_0^1 (g')^2 + \frac{1}{2} \int_0^1 (1-g^2)^2 \geq \Big(\frac{8}{3} - \varepsilon\Big) \frac{1}{\sqrt{\mu}} \tag{2.18}
$$

for any g in D_δ^μ and $\varepsilon = \epsilon(\mu) > 0$ going to zero as $\mu \uparrow \infty$. This converts the problem into one of bounding $\int_0^1 (1-g^2)^2$ above.

Let $f \in D^\mu$. Directly from the definition of D^μ we see that

$$
\int f^4 - 2\Big(\int f^4\Big)^{1/2} + 1 - 2\mu^{-1/2}(\gamma+\eta) \leq 0,
$$

and the inequalities

$$
1 - \sqrt{2(\gamma+\eta)/\sqrt{\mu}} \leq \|f\|_{L^4}^2 \leq 1 + \sqrt{2(\gamma+\eta)/\sqrt{\mu}} \tag{2.19}
$$

follow immediately. Next, if g is to satisfy $\|f-g\|_\infty < \delta$, then $|f+g|^2 < 4|f|^2 + 4\delta|f| + \delta^2$ and so

$$
\int_0^1 |f+g|^2 \leq (\delta + 2\|f\|_{L^4})^2
$$

by Hölder's inequality. This last bound, together with (2.19), implies that

$$
\int_0^1 |f+g|^2 \leq \Bigg(\delta + 2\sqrt{1 + \sqrt{2(\gamma+\eta)\ \sqrt{\mu}}}\Bigg)^2 < 25
$$

for $\mu > 1$ and $\eta, \delta, \gamma < 1$, and so also

$$
\begin{aligned}
\int_0^1 \big(1-g^2\big)^2 - \int_0^1 \big(1-f^2\big)^2 &\leq \delta \int_0^1 |f+g| \big(|1-f^2| + |1-g^2|\big) \\
&\leq 5\delta \left[\sqrt{\int_0^1 (1-f^2)^2} + \sqrt{\int_0^1 (1-g^2)^2}\right].
\end{aligned}
$$

after using Schwarz's inequality. Therefore, $||(1-g^2)||_2 \leq ||(1-f^2)||_2 + 5\delta$ which, after squaring both sides and applying Cauchy's inequality, further implies that

$$
\begin{aligned}
\int_0^1 \big(1-g^2\big)^2 &\leq (1+\eta) \int_0^1 \big(1-f^2\big)^2 + 25\big(1+\eta^{-1}\big)\delta^2 \\
&\leq 2(1+\eta)(\gamma+\eta)\frac{1}{\sqrt{\mu}} + 25\big(1+\eta^{-1}\big)\delta^2
\end{aligned}
$$

for all $g \in D_\delta^\mu$. Put together with (2.18), we have produced

$$\mathcal{I}\Big(D_\delta^\mu(\gamma,\eta)\Big) \geq \left(\frac{8}{3} - \varepsilon - (\gamma+\eta)(1+\eta)\right)\sqrt{\mu} - \frac{25}{2}\left(1+\eta^{-1}\right)\delta^2\mu \tag{2.20}$$

as the desired lower bound.

The final step revisits (2.17) which, with the help of (2.20), says

$$\begin{aligned} P_0^\mu\Big(D^\mu(\gamma,\eta)\Big) \quad &\leq \quad 32\frac{\sqrt{n}}{\delta\sqrt{\mu}}\exp\left[\frac{-\mu n\delta^2}{32}\right] \\ &+ \left(\frac{1}{\sqrt{\eta}}\right)^n \exp\left[\mu^2\,\frac{25}{2}\left(1+\eta^{-1}\right)\delta^2\right]\exp\left[-\mu^{3/2}(1-\eta)\left(\frac{8}{3}-\varepsilon-(\gamma+\eta)(1+\eta)\right)\right]. \end{aligned} \tag{2.21}$$

A careful choice of parameters will now make all terms on the right negligible compared to the one of the form $\exp[-\mu^{3/2} etc.]$. For example, $\delta = \mu^{-\alpha}$ and $n \sim \mu^\beta$ with $\frac{1}{4} < \alpha < \frac{1}{2}$ and $\frac{3}{2} < \beta < \frac{1}{2} - 2\alpha$ (let say $\alpha = 5/16$ and $\beta = 5/4$) will do the job. It follows that

$$\limsup_{\mu\to\infty} \mu^{-3/2}\log P_0^\mu\Big(D^\mu(\gamma,\eta)\Big) \leq -(1-\eta)\left(\frac{8}{3} - (\gamma+\eta)(1+\eta)\right),$$

and so, going back to the original quantity (see (2.10) through (2.12)),

$$\begin{aligned} \limsup_{\mu\to\infty}\ \mu^{-3/2}\log\int_{H^\mu(\gamma,\eta)} e^{-\frac{1}{2}\int_0^1(\mu-p^2)^2}dP_0 \quad &\leq \quad -(1-\eta)\left(\frac{8}{3}-(\gamma+\eta)(1+\eta)\right) + (-\gamma+\eta) \\ &\leq \quad -(1-\eta)\frac{8}{3} + \gamma\eta^2 + \eta\left(2+\eta^2\right) \end{aligned}$$

for any positive $\gamma < 1$. At last, from the decomposition (2.9), we deduce that

$$\limsup_{\mu\to\infty}\ \mu^{-3/2}\log\int_H e^{\frac{1}{2}\int_0^1(\mu-p^2)^2}dP_0 \leq -(1-\eta)\frac{8}{3} + \frac{8}{3}\eta^2 + \eta\left(2+\eta^2\right)$$

and letting $\eta \downarrow 0$ completes the proof.

Proof of Theorem 2.3 We begin by following the blueprint of the proof just completed. First, as in (2.9), the integral over $H\backslash C_\varepsilon^\mu$ is first overestimated by a sum of integrals according to the inclusion

$$\Big(C_\varepsilon^\mu\Big)^c \subseteq \left\{\bigcup\nolimits_{0\leq k\leq(8/3+\eta)/\zeta}\Big(H^\mu(\zeta k,\zeta)\cap(C_\varepsilon^\mu)^c\Big)\right\}\ \bigcup\ \left\{p : \frac{1}{2}\int_0^1(\mu-p^2)^2 \geq (8/3+\eta)\mu^{3/2}\right\}$$

with any $\zeta > 0$. The integral of $\exp[-(1/2)\int_0^1(\mu-p^2)^2]$ over the last set on the right hand side trivially satisfies the desired bound. Next, as in (2.11) and (2.10), bounding the other integrals in this decomposition comes down to bounding the P_0 probabilities of the events $\{H^\mu(\zeta k,\zeta)\cap(C_\varepsilon^\mu)^c\}$. Following the proof of Theorem 2.2 further we come to the critical point. By comparison with (2.18) and the surrounding discussion we see that we now need an improved version of that variational inequality. In particular, we require a lower bound of the form

$$\frac{1}{2\mu}\int_0^1(g')^2 + \frac{1}{2}\int_0^1(1-g^2)^2 \geq (8/3+\tilde\eta)\frac{1}{\sqrt{\mu}}$$

where $\tilde\eta = \tilde\eta(\varepsilon) > 0$ depends on ε but is fixed for all g restricted to lie in $(C_\varepsilon^\mu)^c$, appropriately scaled and δ-enlarged. With the appropriate scaling, it is equivalent to show that

$$\liminf_{a\to\infty}\left(\inf\Big\{I(f;a),\ f\in H_a\cap d(f,\{f_a^*\})>\varepsilon\Big\}\right) > 8/3 \tag{2.22}$$

where $I(f;a)$ and H_a are as defined in (2.2) and $\{f_a^*\}$ represents the set of minimizers of $I(f;a)$.

We argue by contradiction. If (2.22) failed to hold, we could find a sequence $\{\tilde{f}_a\}$ satisfying the constraints, but so that $I(\tilde{f}_a;a)\rightarrow 8/3$. By virtue of the fact that $\int_{-a}^{a}\tilde{f}_a = 0$, each $\tilde{f}_a$ has at least two zeros, $z_a^1 < z_a^2$. Further, $|z_a^1 - z_a^2|\rightarrow\infty$ and $(2a - |z_a^1 - z_a^2|)\rightarrow\infty$. If instead all the zeros were contained in a fixed interval I (which may be assumed to be centered about the origin), it would follow that $|\int_{I^c}\tilde{f}_a|$ must exceed a positive multiple of a. Indeed, $\int_{-a}^{a}[1-(\tilde{f}_a)^2]^2$ remains bounded and the length of I^c is itself $O(a)$. But then $|\int_I \tilde{f}_a| \geq \mathit{const.}\times a$ to maintain the mean zero condition causing $|\tilde{f}_a| \geq \mathit{const.}\times a$ on some subset of I of positive measure. This in turn would imply that $\int_I[1-(\tilde{f}_a)^2]^2$ grows without bound as $a\rightarrow\infty$, and that is impossible.

At this point we return to the strategy behind the proof of Theorem 2.1. First we fix large symmetric intervals of length $2M$ around each of the two zeros z_a^1 and z_a^2 specified thus far. When considering these intervals separately we will identify z_a^1 or z_a^2 with the origin as may be done by translation. By the core argument behind the proof of Theorem 2.1, on each of these intervals we can find subsequences $\{\tilde{f}_a^1\}$ and $\{\tilde{f}_a^1\}$ converging to f_∞^1 and f_∞^2 respectively. Again, both f_∞^1 and f_∞^2 lie in $H^1\cap L^\infty$. Also, since both converge weakly in H^1, by lower semi-continuity of the functional I, it follows that $I(f_\infty^k,\infty)\geq 4/3$, and both also satisfy the equation $f_\infty' = \pm(1-f_\infty^2)$ on their respective domains. That is, $f_\infty^k(x) = \pm\tanh(x)$ for $k=1,2$.

Next we show that z_a^1 and z_a^2 are in fact the only zeros of $\tilde{f}_a$ for $a\rightarrow\infty$, and that $z_a^2 - z_a^1$ is roughly a:

$$\lim_{a\uparrow\infty}\Big(|z_a^1 - z_a^2| - a\Big) = 0. \tag{2.23}$$

For the first part, simply note that if there were a third zero we may repeat the argument of the preceding paragraph to conclude that the rate function I would exceed $4/3+4/3+4/3$ as $a\rightarrow\infty$, but that contradictions the assumption $I(\tilde{f}_a,a)\rightarrow 8/3$. For the statement regarding the distance between z_a^1 and z_a^2, note first that the integrals of $\{\tilde{f}_a\}$ over $[(z_a^1+z_a^2)/2,(z_a^1+z_a^2)/2-a]$ or over its complement in $[-a,a)$ must have the same absolute value but opposite signs. Translating to place z_1^a at the origin we find that

$$\int_\ell^u \tilde{f}_a(x)dx + \int_{-u}^{-\ell}\tilde{f}_a(x)dx = 0$$

where $u=(z_a^2-z_a^1)/2$ and $\ell=(z_a^2-z_a^1)/2-a$. We can now assume that $f_\infty^1=\tanh$ so that $f_\infty^2=-\tanh$. Then, since there is uniform convergence to these limiting functions, both integrals in the last display are approximately $u+\ell$. It follows that $u+\ell\rightarrow 0$ as $a\rightarrow\infty$ which is the same as (2.23).

The conclusion is that $||\tilde{f}_a - f_a||_\infty\rightarrow 0$ where f_a is the test function constructed in (2.3). Since f_a minimizes $I(f;a)$ as $a\rightarrow\infty$ we have shown that $d(\tilde{f}_a,\{f_a^*\})\rightarrow 0$, contradicting the original hypothesis. The proof is complete.

3 Expanding about the extrema

Following the classical Laplace method we wish to expand $f(\mu)$ in the vicinity of each I_μ-minimizer. While we have not actually computed any such minimizer at finite μ (nor have we proved the anticipated uniqueness up to translation), it suffices to introduce the following proxy. The Euler-Lagrange equation (2.4), describing the scaled minimizer(s), may be solved in terms of Jacobi elliptic functions. Thus motivated, we bring in

$$p_\mu(x) = k\sqrt{\mu}\times\mathrm{sn}(\sqrt{\mu}x,k) \tag{3.1}$$

along with its translates $p_\mu^a(\cdot) = p_\mu(\cdot + a)$. As a function of the real variable x, $\mathrm{sn}(x,k)$ is periodic with period determined by its *modulus* $k\in[0,1]$. In particular, $\mathrm{sn}(\cdot,k)=\mathrm{sn}(\cdot+4K,k)$ in which K is the complete elliptic integral of the first kind: $K(k)=\int_0^1[(1-x^2)(1-k^2x^2)]^{-1/2}dx$. (For background on *sin-amp* and other elliptic functions used throughout, [4] is recommended.)

In p_μ, we must choose k so that $4K = \sqrt{\mu}$, and it may be deduced that $k^2 \simeq 1 - 16e^{-\sqrt{\mu}/2}$. With this parameter set, an exact computation will yield

$$I_\mu(p_\mu) = \frac{8}{3}\mu^{3/2} + O\Big(e^{-\sqrt{\mu}/4}\Big). \tag{3.2}$$

While this is certainly heart-warming, a more important connection with the discussion in Section 2.1 is seen in the fact that $\mathrm{sn}(x,k) \simeq \tanh(x)$ for $k\uparrow 1$. So, with p_μ^* any I_μ-minimizer with $p_\mu^*(0) = 0$, the proofs of Theorem 2.1 and 2.3 will explain why

$$\lim_{\mu\to\infty} \Big|\Big| \frac{1}{\sqrt{\mu}} p_\mu - \frac{1}{\sqrt{\mu}} p_\mu^* \Big|\Big|_\infty = 0.$$

In other words, an appropriate L^∞ tube about the set of translates $\{p_\mu^a\}$ contains a like tube about the set of I_μ-minimizers. Theorem 2.3 then implies the following.

Corollary 3.1 *Let $\varepsilon > 0$. Then*

$$f(\mu) = \frac{1}{\sqrt{2\pi}} E_0\Big[e^{-\frac{1}{2}\int_0^1 (\mu - p^2)^2} A(p),\ d(p, \{p_\mu^a\}) \le \varepsilon\sqrt{\mu}\Big] + O\Big(e^{-(8/3+\eta)\mu^{3/2}}\Big), \tag{3.3}$$

with some $\eta = \eta(\varepsilon) > 0$.

This last observation turns the problem of expanding $f(\mu)$ about each I_μ-minimizer into that of expanding the expectation in (3.3) about each p_μ^a, $0 \le a < 1$. While a definite advancement, we must still confront the degeneracy inherent in the translation invariance of I_μ. We handle this issue a conditioning procedure in order to pin the E_0 expectation about a single p_μ^a, which for convenience is taken to be $p_\mu = p_\mu^0$. The idea is that since the translations $p_\mu^0 \to p_\mu^{0+\varepsilon}$ are generated by $p_\mu' = const. \times \mathrm{cn}(\sqrt{\mu}x, k)\mathrm{dn}(\sqrt{\mu}x, k)$, conditioning the path to be orthogonal to p_μ' will keep the path in a small neighborhood of $\pm p_\mu$ as opposed to translates further afield. Relating the E_0 expectation to this conditioned version of itself requires a change of measure formula provided in the following lemma; the proof is deferred to the appendix.

Lemma 3.1 *Suppose $X(\cdot)$ is a smooth stationary process, periodic, of period one. Also assume that, with probability one, X has at least one zero. If F is a functional that is invariant under translations of the path, then*

$$E\Big[F(X)\Big] = E\Big[F(X)\frac{1}{\mathcal{N}^{-1}}\Big| X(0) = 0\Big] P\Big(X(0) = 0\Big)$$

where

$$\mathcal{N} = \mathcal{N}(X) = \sum_{z\in\mathcal{Z}} |X_z'|^{-1}$$

and $\mathcal{Z}$ is the set of zeros of X. Here and throughout, the notation $P(X = a)$ indicates the density of X at a.

Applying Lemma 3.1 to the matter at hand, we set[1]

$$\phi_1^\mu(x) \equiv \frac{\mathrm{cn}(\sqrt{\mu}x, k)\mathrm{dn}(\sqrt{\mu}x, k)}{\sqrt{\int_0^1 \mathrm{cn}(\sqrt{\mu}x', k)\mathrm{dn}(\sqrt{\mu}x', k)dx'}},$$

and note the following.

[1]The choice of notation will become clear in the next section.

Corollary 3.2 *Let*

$$R(p) \equiv \mathcal{N}^{-1}\left(x \to \int_0^1 \phi_1^\mu(x+x')p(x')dx'\right), \tag{3.4}$$

then

$$\begin{aligned}&E_0\left[e^{-\frac{1}{2}\int_0^1(\mu-p^2)^2}A(p), d(p,\{p_\mu^a\}) \le \varepsilon\sqrt{\mu}\right] \\ &\quad = \quad E_0^0\left[e^{-\frac{1}{2}\int_0^1(\mu-p^2)^2}A(p)R(p), d(p,\{p_\mu^a\}) \le \varepsilon\sqrt{\mu}\right]\ P_0\Big(\int_0^1 \phi_1^\mu p = 0\Big).\end{aligned}$$

Here E_0^0 is now the CBM conditioned so that both $\int_0^1 p = 0$ and $\int_0^1 \phi_1^\mu p = 0$.

Next, consider the intersection of $\{p : d(p,\{p_\mu^a\}) \le \varepsilon\sqrt{\mu}\}$ and $\{p : \int_0^1 \phi_1^\mu p = 0\}$. It is easy to see that the resulting set will contain the union of $\{||p - p_\mu^0||_\infty \le \varepsilon_1\sqrt{\mu}\}$ and the same object with p_μ^0 replaced by $p_\mu^{1/2} = -p_\mu^0$ for some ε_1 small enough. Likewise, it will be contained in a similar union with ε_1 replaced by some larger $\varepsilon_2 > 0$. Of course, the integral in question is invariant under the sign change $p \to -p$. These comments along with Corollaries 3.2 and 3.1 allow the following statement:

$$f(\mu) \simeq \sqrt{\frac{2}{\pi}}E_0^0\left[e^{-\frac{1}{2}\int_0^1(\mu-p^2)^2}A(p)R(p), ||p-p_\mu||_\infty \le \varepsilon\sqrt{\mu}\right]P_0\Big(\int_0^1 \phi_1^\mu p = 0\Big) \tag{3.5}$$

up to $O(e^{-(8/3+)\mu^{3/2}})$ errors granted that we eventually prove that the desired level of asymptotics for the E_0^0 integral in (3.5) are independent of $\varepsilon > 0$.

Having centered the integral about the single path p_μ, we complete this section by performing the change of variables $p \to p + p_\mu$ in order to bring the contribution of p_μ up into the exponent.

Proposition 3.1 *With the E_0^0 integral on the right hand side of (3.5) denoted by $f(\mu;\varepsilon)$ we have*

$$f(\mu;\varepsilon) = e^{-I_\mu(p_\mu)}E_0^0\left[e^{-\frac{1}{2}\int_0^1(q_\mu-2\mu)p^2}e^{-2p_\mu\int_0^1 p_\mu p^3 - \frac{1}{2}\int_0^1 p^4}A(p+p_\mu)R(p+p_\mu), ||p||_\infty \le \varepsilon\sqrt{\mu}\right] \tag{3.6}$$

in which $q_\mu(x) \equiv 6\mu k^2 sn^2(\sqrt{\mu}x, k)$ $(= 6p_\mu^2(x))$.

We have thus extracted the advertised leading term, $e^{-I_\mu(p_\mu)} = e^{-8/3\mu^{3/2}}(1+o(1))$. Further, the formula (3.6) identifies the Gaussian measure $e^{-(1/2)\int_0^1(q_\mu-2\mu)p^2}dP_0^0$ which, as is the case in finite dimensional Laplace asymptotics, will dictate the remainder of our computation. The study of this measure is initiated in the next section.

Remark Given (3.6), it is a simple matter to obtain the lower bound complementing Theorem 2.2: $\lim_{\mu\to\infty}\mu^{-3/2}\log f(\mu) = -8/3$.

Remark If we now understand that a vicinity of p_μ (and its translates) accounts for the leading order behavior of $f(\mu)$ for $\mu \to \infty$, it is interesting to consider what this implies for the random potential. Running the Ricatti correspondence "backwards" relates this leading path to the potential $q(x;\mu) \equiv -\mu + p_\mu'(x) + p_\mu^2(x) \simeq -2\mu\,\mathrm{sech}^2(\sqrt{\mu}(x-1/2))$. While formal, this indicates that large negative deviations of the ground state stem from White-Noise potentials lying nearby a single well of depth μ and width $1/\sqrt{\mu}$.

Proof of Proposition 3.1 This is a consequence of the Cameron-Martin formula for P_0^0 proved in the Appendix (Lemma 8.2). It states that, for bounded functions F of the path,

$$E_0^0[F(p)] = E_0^0\left[F(p+p_\mu)\exp\left\{\int_0^1 p_\mu'' p - \frac{1}{2}\int_0^1 |p_\mu'|^2\right\}\right].$$

In the present case

$$F(p) = e^{-\frac{1}{2}\int_0^1(\mu-p^2)^2} A(p)R(p)1_{\{||p-p_\mu||_\infty \le \varepsilon\sqrt{\mu}\}},$$

and a simple expansion yields

$$\begin{aligned} &\frac{1}{2}\int_0^1 \Big(\mu - (p+p_\mu)^2\Big)^2 + \frac{1}{2}\int_0^1 |p_\mu'|^2 - \int_0^1 p_\mu'' p \\ &= \quad I_\mu(p_\mu) + 2\int_0^1 p_\mu(p_\mu^2-\mu)p + \frac{1}{2}\int_0^1 \Big(6p_\mu^2 - 2\mu + 4p_\mu p + p^2\Big)p^2 - \int_0^1 p_\mu'' p. \end{aligned}$$

Next, we notice that $p_\mu'' = 2p_\mu^3 - 2\mu p_\mu + constant$, and therefore

$$-\int_0^1 p_\mu'' p = -2\int_0^1 p_\mu(p_\mu^2-\mu)p$$

when $\int_0^1 p = 0$. The proof is finished by combining the last two formulas.

4 Hill's Spectrum and the associated Gaussian process

Our analysis of the culminating form of the expectation (3.6) takes the anticipated route. Restricted to a set of relatively small L^∞ norm, it is expected that $A(\cdot)$ and $R(\cdot)$ settle down to $A(p_\mu)$ and $R(p_\mu)$ as $\mu\to\infty$. More delicate, the cubic ($2\int_0^1 p_\mu p^3$) and quartic ($-(1/2)\int_0^1 p^4$) terms in the exponential should be lower order when compared with the quadratic factor $(1/2)\int_0^1(q_\mu - 2\mu)p^2$ which is of order μ. That is, the remainder of the computation should be viewed with respect to the Gaussian measure $\exp[-(1/2)\int_0^1(q_\mu-2\mu)p^2]dP_0^0$ arising from the Hessian of the rate function $I_\mu(p)$ at $p = p_\mu$.

Of course, exercising this program first requires being able to compute in the latter measure. This prompts the study of the periodic spectrum of the (deterministic) operator

$$Q_\mu = -\frac{d^2}{dx^2} + q_\mu(x),$$

and in this we are met with no small piece of good fortune. The reason we can compute the details of $f(\mu)$ rests on the rather special properties of Q_μ.

4.1 Hill's equations

Consider the general Hill's operator $Q = -d^2/dx^2 + q(x)$ in which $q(x)$ is smooth and with period now taken to be 1/2 (note our motivating example Q_μ). The periodic spectral points of Q comprise a list:

$$-\infty < \lambda_0 < \lambda_1 \le \lambda_2 < \lambda_3 \le \lambda_4 < \lambda_5 \le \lambda_6 < \lambda_7 \le \cdots \uparrow +\infty.$$

In particular, the so-called principal series $\lambda_0 < \lambda_3 \le \lambda_4 < \lambda_7 \le \lambda_8 < \cdots$ makes up the periodic spectrum of Q acting on L^2 functions of period 1/2, and the complementary series $\lambda_1 \le \lambda_2 < \lambda_5 \le \lambda_6 < \cdots$ fills out the periodic spectrum of Q on L^2 functions of period 1. An equivalent characterization of spectrum may be described with the help of Hill's discriminant

$$\Delta(\lambda) = y_1(1/2,\lambda) + y_2'(1/2,\lambda)$$

in which $y_1(1/2,\lambda)$ ($y_2(1/2,\lambda)$) is the normalized sine-like (cosine-like) solution of $Qy = \lambda y$ with $y(0) = 0, y'(0) = 1$ ($y(0) = 1$, $y'(0) = 0$). The classical result is that $\Delta(\lambda)$ is an entire function of order 1/2 and that it encodes the spectrum: $\Delta(\lambda) = +2$ on the principal series and $\Delta(\lambda) = -2$ on the complementary series.

A special situation occurs when the shape of the potential q is such that the simple eigenvalues $\lambda_0 < \lambda_1 < \cdots < \lambda_{2g}$ are finite in number with the rest of the list double: $\lambda_{2\ell-1} = \lambda_{2\ell}$ for $\ell > g$. Then Q is said to be *finite gap*, and it is the remarkable discovery of Hochstadt [10] that in this case the simple spectrum determines the full spectrum and so also $\Delta(\lambda)$. More precisely:

Hochstadt's Formula *Let Q be finite-gap with $2g+1$ simple eigenvalues. Then $\Delta(\lambda) = 2\cos\psi(\lambda)$ with*

$$\psi(\lambda) = \frac{\sqrt{-1}}{2}\int_{\lambda_0}^{\lambda} \frac{(s-\lambda_1')\cdots(s-\lambda_g')}{\sqrt{-(s-\lambda_0)\cdots(s-\lambda_{2g})}}ds \tag{4.1}$$

in which $\lambda_1' < \cdots < \lambda_g'$ are the points $\lambda_{2\ell-1} < \lambda_\ell' < \lambda_{2\ell}$ where $\Delta'(\lambda) = 0$. They are determined from the simple spectrum through the requirement: $\psi(\lambda_{2\ell}) - \psi(\lambda_{2\ell-1}) = 0$ for $\ell = 1, 2, \ldots g$.

This formula will play an essential role in Sections 6 and 7. More background information on Hill's spectrum can be found in [18] or [21].

4.2 When $Q = Q_\mu$

The key fact is that the family of Lamé operators $Q = -d^2/dx^2 + m(m+1)k^2\mathrm{sn}^2(x,k)$ over the period $0 \le x \le 2K$ are finite gap with $g = m$ (see again [21]). In Q_μ we have $m = 2$, and, what is more, the simple eigenvalues and corresponding eigenfunctions are known, having first been computed in [12]. With

$$a_\pm(k) = 1 + k^2 \pm \sqrt{1 - k^2 + k^4},$$

we have:

$$\begin{aligned} \lambda_0 &= 2a_-(k) & \tilde\phi_0(x) &= 1 - a_-(k)\mathrm{sn}^2(x,k) \\ \lambda_1 &= 1 + k^2 & \tilde\phi_1(x) &= \mathrm{cn}(x,k)\mathrm{dn}(x,k) \\ \lambda_2 &= 1 + 4k^2 & \tilde\phi_2(x) &= \mathrm{sn}(x,k)\mathrm{dn}(x,k) \\ \lambda_3 &= 4 + k^2 & \tilde\phi_3(x) &= \mathrm{sn}(x,k)\mathrm{cn}(x,k) \\ \lambda_4 &= 2a_+(k) & \tilde\phi_4(x) &= 1 - a_+(k)\mathrm{sn}^2(x,k). \end{aligned} \tag{4.2}$$

Of course, things are to be scaled as in $x \to 4Kx = \sqrt{\mu}x$ to keep the period at 1/2. Accordingly, $\lambda_\ell \to \lambda_\ell^\mu = \mu \times \lambda_\ell$ and $\tilde\phi_\ell(x) \to \tilde\phi_\ell^\mu(x) = \tilde\phi_\ell(\sqrt{\mu}x)$, after which we may introduce the $L^2[0,1]$-orthonormal sequence denoted by $\{\phi_\ell^\mu\} = \{\tilde\phi_\ell^\mu/||\tilde\phi_\ell^\mu||_2\}$. Keep in mind though the unscaled λ's and $\tilde\phi$'s depend on μ through the modulus k. Further, it is worthwhile noting that unscaled operator $\mu^{-1/2}Q_\mu(\cdot/\sqrt{\mu})$ tends to $-d^2/dx^2 + 6\tanh^2(x)$ over the whole line as $\mu \to \infty$ or $k \to 1$. In this picture, $\tilde\phi_0, \ldots, \tilde\phi_3$ correspond to bound states ($\tilde\phi_0, \tilde\phi_1 \simeq \mathrm{sech}^2$ and $\tilde\phi_2, \tilde\phi_3 \simeq \sinh \mathrm{sech}^2$ up to $O(e^{-\sqrt{\mu}/2})$ errors in L^∞) with continuous spectrum beginning at $\lambda_4 \simeq 6$.

4.3 The Gaussian Measure

We now have a more complete picture connecting the basic degeneracy and the introduced conditioning. If the CBM had appeared unconditional in the definition of $f(\mu)$, we would at this point be faced with the measure $\exp[-1/2\int_0^1(q_\mu - 2\mu)p^2] \times dCBM$ to provide concentration of the path about the extrema set. However, the first two members of the corresponding spectrum $\lambda_0^\mu - 2\mu$ and $\lambda_1^\mu - 2\mu$ are $\simeq -16e^{-\sqrt{\mu}/2}$, preventing this measure from having a sense.[2] Thus, the mean-zero conditioning built into the problem works to counteract the first degeneracy tied to the ground state of Q_μ. Furthermore, the conditioning we introduced to account for the translational degeneracy exactly removes the second eigenstate of Q_μ.

[2] That these first two spectral points are even a little negative is due to the fact that p_μ is not actually an extrema, though exponentially close.

That the imposed $\int_0^1 p = 0$ only "works to counteract" the degeneracy at ϕ_0^μ is because that mode is not fully removed by the conditioning. The constant function can however be written in the eigenbasis: from the first and last items of the list (4.2) we see that

$$1 = c_0 \phi_0^\mu(x) - c_4 \phi_4^\mu(x) \tag{4.3}$$

where

$$c_0 = c_0(\mu) = \Big(\frac{a_+(k)}{a_+(k) - a_-(k)} \Big) ||\tilde{\phi}_0^\mu||_2 \quad \text{and} \quad c_4 = c_4(\mu) = \Big(\frac{a_-(k)}{a_+(k) - a_-(k)} \Big) ||\tilde{\phi}_4^\mu||_2. \tag{4.4}$$

This allows an explicit description of the bottom of the conditioned spectrum.

Lemma 4.1 *The Gaussian measure P_μ^* defined by*

$$E_\mu^* \Big[F(p) \Big] = \frac{E_0^0 \Big[F(p) \exp \Big\{ -\frac{1}{2} \int_0^1 (q_\mu(x) - 2\mu) p^2(x) dx \Big\} \Big]}{E_0^0 \Big[\exp \Big\{ -\frac{1}{2} \int_0^1 (q_\mu(x) - 2\mu) p^2(x) dx \Big\} \Big]}$$

has the expansion

$$\begin{aligned} p(x) \;\; &= \;\; \Big(\frac{c_4 \phi_0^\mu(x) + c_0 \phi_4^\mu(x)}{\sqrt{c_4^2(\lambda_0^\mu - 2\mu) + c_0^2(\lambda_4^\mu - 2\mu)}} \mathfrak{g}_0 + \frac{\phi_2^\mu(x)}{\sqrt{\lambda_2^\mu - 2\mu}} \mathfrak{g}_2 + \frac{\phi_3^\mu(x)}{\sqrt{\lambda_3^\mu - 2\mu}} \mathfrak{g}_3 \Big) \\ &\quad + \Big(\sum_{\ell \geq 5} \frac{1}{\sqrt{\lambda_\ell^\mu - 2\mu}} \phi_\ell^\mu(x) \mathfrak{g}_\ell \Big) \\ &\equiv \;\; p_\ell(x) \; + \; p_h(x) \end{aligned} \tag{4.5}$$

for $\{\mathfrak{g}_\ell\}$ a sequence of independent standard Gaussians.

The division into low and high modes (p_ℓ and p_h) is prepared for later. We pause here to note that while the typical normalizer of each high mode $\sqrt{\lambda_\ell^\mu - 2\mu}$ is $O(\sqrt{\mu})$ (for any $\ell \geq 2$), the corresponding object in the lowest mode $\sqrt{c_0^2(\lambda_4^\mu - 2\mu) + etc.}$ is only of order $\mu^{1/4}$. Thus, scaling out the $\sqrt{\mu}$ from the exponential weight of P_μ^* (as is customary in Laplace type computations), we see that the resulting Gaussian measure has a spectral gap disappearing like $\mu^{-1/4}$. This is degeneracy (or perhaps it is better to say "near-degeneracy") orthogonal to the set of extrema mentioned in the introduction. It would be interesting to understand its physical significance.

Proof This is just the classical Karhunen-Loève expansion, see [1]; we outline the derivation to make clear what occurs at the bottom of the spectrum on account of the condition $\int_0^1 p = 0$.

As $CBM[\exp\{-1/2 \int_0^1 (q_\mu - 2\mu) p^2\}] = +\infty$, it is convenient to express things through $dP_{q_\mu} = Z_{q_\mu}^{-1} \exp[-1/2 \int_0^1 q_\mu p^2] dCBM$ which does have finite total mass. That is, we write

$$E_\mu^* \Big[F(p) \Big] = \frac{E_{q_\mu} \Big[F(p) \, e^{\mu \int_0^1 p^2(x) dx} \, \Big| \, \int_0^1 p(x) = 0, \int_0^1 p(x) \phi_1^\mu(x) = 0 \Big]}{E_{q_\mu} \Big[e^{\mu \int_0^1 p^2(x) dx} \, \Big| \, \int_0^1 p(x) = 0, \int_0^1 p(x) \phi_1^\mu(x) = 0 \Big]}, \tag{4.6}$$

and introduce the coordinates $p(x) = \sum_{\ell=0}^\infty \frac{1}{\sqrt{\lambda_\ell^\mu}} \phi_\ell^\mu(x) \eta_\ell$ under P_{q_μ} (the η_ℓ's are independent standard Gaussians). Written in this way, the numerator of (4.6) takes the form

$$\begin{aligned} & E \Big[F \Big(\sum_{\ell=0}^\infty \frac{1}{\sqrt{\lambda_\ell^\mu}} \phi_\ell^\mu(x) \eta_\ell \Big) \, \exp \Big\{ \sum_{\ell=0}^\infty \frac{\mu}{\lambda_\ell^\mu} \eta_\ell^2 \Big\} \, \Big| \, \frac{c_0}{\sqrt{\lambda_0^\mu}} \eta_0 - \frac{c_4}{\sqrt{\lambda_4^\mu}} \eta_4 = 0, \, \eta_1 = 0 \Big] \\ & \quad = \;\; C \, E \Big[F \Big(\frac{1}{\sqrt{\lambda_0^\mu}} \phi_0^\mu(x) \eta_0 + \frac{1}{\sqrt{\lambda_4^\mu}} \phi_4^\mu(x) \eta_4 + \tilde{p}_h(x) \Big) \exp \Big\{ \frac{\mu}{\lambda_4^\mu} \eta_1^2 + \frac{\mu}{\lambda_4^\mu} \eta_4^2 \Big\} \Big| \, \frac{c_0}{\sqrt{\lambda_0^\mu}} \eta_0 - \frac{c_4}{\sqrt{\lambda_4^\mu}} \eta_4 = 0 \Big]. \end{aligned}$$

Here C is a constant factor and

$$\tilde{p}_h(x) = \frac{\phi_2^\mu(x)}{\sqrt{\lambda_2^\mu - 2\mu}}\mathfrak{g}_2 + \frac{\phi_3^\mu(x)}{\sqrt{\lambda_3^\mu - 2\mu}}\mathfrak{g}_3 + p_h(x),$$

its distribution now identified by independence.

There remains the distribution of $p_{0,4}(x) = \frac{1}{\sqrt{\lambda_0^\mu}}\phi_0^\mu(x)\eta_0 + \frac{1}{\sqrt{\lambda_4^\mu}}\phi_4^\mu(x)\eta_4$ subject to the quadratic weight $\exp[\frac{\mu}{\lambda_4^\mu}\eta_1^2 + \frac{\mu}{\lambda_4^\mu}\eta_4^2]$ and linear conditioning $\{\frac{c_0}{\sqrt{\lambda_0^\mu}}\eta_0 - \frac{c_4}{\sqrt{\lambda_4^\mu}}\eta_4 = 0\}$. The question is only whether the latter conditioning can hold down the focusing weight. A straightforward computation shows that, for $X, Y \sim N(0,1)$ and independent,

$$(1-2\alpha) + \Big(\frac{a}{b}\Big)^2(1-2\beta) > 0 \quad \text{implies} \quad E[e^{\alpha X^2 + \beta Y^2} \,|\, aX + bY = 0] < \infty.$$

Next note that, by inspection, $a_+(k) \simeq 3$ and $a_-(k) \simeq 1$ up to errors of order $1 - k^2 = O(e^{-\sqrt{\mu}/2})$. At the same level approximation we also have

$$||\tilde{\phi}_0||_2^2 \simeq \frac{1}{\sqrt{\mu}}\int_0^{\sqrt{\mu}} \mathrm{cn}^4(x,k)dx \simeq \frac{2}{\sqrt{\mu}}\int_{-\infty}^{\infty} \mathrm{sech}^4(x)dx = \frac{8}{3\sqrt{\mu}},$$

and $||\tilde{\phi}_4||_2^2 \simeq 4$; the upshot being that $c_0(\mu) \simeq \sqrt{6}\mu^{-1/4}$ and $c_4(\mu) \simeq 1$. Thus,

$$\Big(1 - \frac{2\mu}{\lambda_0^\mu}\Big) + \Big(\frac{c_0^2\lambda_4^\mu}{c_4^2\lambda_0^\mu}\Big)\Big(1 - \frac{2\mu}{\lambda_4^\mu}\Big) = 12\frac{1}{\sqrt{\mu}} + O(e^{-\sqrt{\mu}/2}),$$

and we just get by. Running through the same computation with the addition of a test function of $p_{0,4}$ in the integrand completes the proof.

5 Proof of the main error estimate

Having renormalized in terms of the Gaussian measure P_μ^* defined in Lemma 4.1, we now return to the asymptotics of $f(\mu,\varepsilon)$, recall (3.6). Using the form of $A(\cdot)$ we first write

$$\begin{aligned} &\sqrt{\frac{\pi}{2}}\frac{1}{Z_\mu^*}e^{-I_\mu(p_\mu)}f(\mu;\varepsilon) \qquad (5.1)\\ &= \int_0^1\int_0^1 e^{2\int_y^x p_\mu} E_\mu^*\Big[e^{2\int_x^y p - 2\int_0^1 p_\mu p^3 - \frac{1}{2}\int_0^1 p^4} R(p+p_\mu),\ ||p||_\infty \le \varepsilon\sqrt{\mu}\Big]\, dy\, dx \end{aligned}$$

in which Z_μ^* is the P_μ^* mass, $Z_\mu^* = E_0^0[\exp\{-\frac{1}{2}\int_0^1(q_\mu - 2\mu)p^2\}]$. In these terms, the main result of this section is that right hand side of (5.1) equals $A(p_\mu)R(p_\mu)$ up to small multiplicative errors, and, since $A(p_\mu) = \int_0^1\int_0^1 e^{2\int_y^x p_\mu}dxdy$, this is equivalent to the following.

Theorem 5.1 *For $\mu \to \infty$ and all $\varepsilon > 0$ sufficiently small,*

$$E_\mu^*\Big[\exp\Big\{2\int_x^y p - 2\int_0^1 p_\mu p^3 - \frac{1}{2}\int_0^1 p^4\Big\}R(p+p_\mu),\ ||p||_\infty \le \varepsilon\sqrt{\mu}\Big] = R(p_\mu)(1+o(1))$$

independently of $x, y \in [0,1]$.

This leaves the computation of Z_μ^*, taken up in the next section, as the final ingredient in the proof of Theorem 1.1.

As alluded to earlier, the fact that the (scaled) process P_μ^* does not posses a good spectral gap means we have to take extra care in dealing with the low modes. In particular, everywhere p is split ($p = p_\ell + p_h$) and expanded, with differing considerations for each piece. To explain some of the difficulty, we introduce the shorthand

$$p_\ell(x) \equiv a_\mu(x)\, \mathfrak{g}_0 + b_\mu(x)\, \mathfrak{g}_2 + c_\mu(x)\, \mathfrak{g}_3$$

and note that for large μ, $a_\mu(x) \simeq (1/8)\mathrm{cn}^2(\sqrt{\mu}x) + O(\mu^{-1/2})$, $b_\mu(x) \simeq (1/2)\mu^{-1/4}\mathrm{sn}(\sqrt{\mu}x)\mathrm{dn}(\sqrt{\mu}x)$, and $c_\mu(x) \simeq (1/2)\mu^{-1/4}\mathrm{sn}(\sqrt{\mu}x)\mathrm{cn}(\sqrt{\mu}x)$, see again (4.5). The point is that while any positive moment of $|a_\mu|$ decays like $\mu^{-1/2}$, this mode remains $O(1)$ in sup-norm as $\mu\to\infty$. The decay of b_μ and c_μ is only slightly better, both of order $\mu^{1/4}$ in L^∞ and order $\mu^{-3/4}$ in L^1. It is not surprising then that we have to rely on cancellations in the lower modes. For example, for the cubic we find that

$$\int_0^1 p_\mu p_\ell^3 = 3\Big(\int_0^1 p_\mu a_\mu^2 b_\mu\Big)\, \mathfrak{g}_0^2\mathfrak{g}_2 + \int_0^1 p_\mu (b_\mu \mathfrak{g}_2 + c_\mu \mathfrak{g}_3)^2, \tag{5.2}$$

the potentially troublesome terms $\int_0^1 p_\mu a_\mu^3$ and $\int_0^1 p_\mu a_\mu^2 c_\mu$ both vanishing. Even still, the remaining term $\int_0^1 p_\mu a_\mu^2 b_\mu \mathfrak{g}_0^2\, \mathfrak{g}_2$ cannot be made small on its own. Besides cancellations, we need help from the (negative) quartic.

Towards this end, we make the decomposition

$$2\int_x^y p + 2\int_0^1 p_\mu p^3 - \frac{1}{2}\int_0^1 p^4 \equiv F_0(p,\mu) + F_1(p,\mu)$$

in which

$$F_0(p,\mu) = 6\Big(\int_0^1 p_\mu a_\mu^2 b_\mu \mathfrak{g}_0^2\mathfrak{g}_2\Big) - \frac{1}{2}\int_0^1 \Big(a_\mu^4\mathfrak{g}_0^4 + b_\mu^4\mathfrak{g}_2^4 + c_\mu^4\mathfrak{g}_3^4\Big) - \frac{1}{2}\int_0^1 p_h^4 - \int_0^1 p_\ell^2 p_h^2 \tag{5.3}$$

and

$$\begin{aligned} F_1(p,\mu) \quad = \quad & 2\int_0^1 p_\mu p_h^3 + 6\int_0^1 p_\mu p_\ell p_h^2 + 6\int_0^1 p_\mu p_\ell^2 p_h \\ & +2\int_x^y p - 3\int_0^1 p_\ell^3 p_h - 3\int_0^1 p_\ell p_h^3 \\ & +2\int_0^1 p_\mu(b_\mu\mathfrak{g}_2 + c_\mu\mathfrak{g}_3)^2 - \frac{1}{2}\int_0^1 (p_\ell^4 - a_\mu^4\mathfrak{g}_0^4 - b_\mu^4\mathfrak{g}_2^4 - c_\mu^4\mathfrak{g}_3^4). \end{aligned} \tag{5.4}$$

With this in hand, we note that for any event $\mathcal{A}$ of the path:

$$\begin{aligned} & \Big| E_\mu^*\Big[e^{F_0(p,\mu)+F_1(p,\mu)}\, R(p+p_\mu),\, \mathcal{A}\Big] - R(p_\mu)E_\mu^*\Big[e^{F_0(p,\mu)},\, \mathcal{A}\Big] \Big| \\ \leq \quad & R(p_\mu)E_\mu^*\Big[e^{F_0(p,\mu)}|F_1(p,\mu)|e^{|F_1(p,\mu)|},\, \mathcal{A}\Big] + E_\mu^*\Big[|R(p_\mu) - R(p+p_\mu)|e^{F_0(p,\mu)},\, \mathcal{A}\Big] \\ & + E_\mu^*\Big[|R(p_\mu) - R(p+p_\mu)|e^{F_0(p,\mu)}\, |F_1(p,\mu)|e^{|F_1(p,\mu)|},\, \mathcal{A}\Big] \end{aligned}$$

where the elementary inequality $|1 - e^f| \leq |f|e^{|f|}$ has been used. Thus, after successive applications of Hölder's inequality, the theorem is a consequence of the following three facts.

Lemma 5.1 *There exists a $\theta' > 1$ such that*

$$E_\mu^*\Big[\exp\Big\{\theta\, F_0(p,\mu)\Big\},\, ||p||_\infty \leq \varepsilon\sqrt{\mu}\Big] = 1 + O(\mu^{-1/2})$$

for all $\theta \in [1,\theta')$ as $\mu\to\infty$.

Lemma 5.2 *Given $\theta > 1$,*

$$\lim_{\mu\to\infty} E^*_\mu\Big[|F_1(p,\mu)|^\theta \, \exp\Big\{\theta|F_1(p,\mu)|\Big\}, \, ||p||_\infty \le \varepsilon\sqrt{\mu}\Big] = 0.$$

for all $\varepsilon > 0$ small enough.

Lemma 5.3 *For any $\theta > 1$ there is the bound*

$$\Big(E^*_\mu\Big[\Big|R(p_\mu + p) - R(p_\mu)\Big|^\theta, \, ||p||_\infty \le \varepsilon\sqrt{\mu}\Big]\Big)^{1/\theta} \le C_0 \, R(p_\mu) \, \mu^{-3/4}$$

in which C_0 depends on ε but not μ.

Furthermore, the proofs of these three ingredients rely to varying degree on the next lemma which describes the convergence of P^*_μ to the zero path as $\mu\to\infty$.

Lemma 5.4 *Under P^*_μ the path has the following decay in mean-square. We have*

$$E^*_\mu\Big[\int_0^1 p^2(x)dx\Big] \le C_1 \, \mu^{-1/2},$$

and, if we remove the low modes:

$$\sup_{0\le x\le 1} E^*_\mu\Big[p^2_h(x)\Big] \le C_2 \, \mu^{-1/2}.$$

Proof For the integrated mean-square estimate over the full path p one just computes:

$$E^*_\mu\Big[\int_0^1 p^2(x)dx\Big] = \int_0^1 a^2_\mu(x)dx + \frac{1}{\lambda^\mu_2 - 2\mu} + \frac{1}{\lambda^\mu_3 - 2\mu} + \sum_{\ell=5}^\infty \frac{1}{\lambda^\mu_\ell - 2\mu}. \tag{5.5}$$

The first three terms we know explicitly. The $\mu^{-1/2}$ decay of $\int_0^1 a^2_\mu$ has already been remarked upon, and both λ^μ_2 and λ^μ_3 are approximately 5μ for μ large. We also know that $\lambda^\mu_\ell - 2\mu = O(\mu)$ for any fixed $\ell \ge 2$ and $\mu\to\infty$. However, controlling the whole sum requires more precise eigenvalue asymptotics provided by the following classical result:

$$\lambda^\mu_\ell = 4\pi^2\ell^2 + \int_0^1 q_\mu + O\Big(\frac{1}{4\pi^2\ell^2}\int_0^1 (q_\mu - \int_0^1 q_\mu)^2\Big) = 4\pi^2\ell^2 + 6\mu + O\Big(\mu\ell^{-2}\Big) \tag{5.6}$$

for all large values of the index. See, for example, Theorem 2.12 of [18]. The tail of the sum in (5.5) then behaves like

$$\sum_{\ell\ge L} \frac{1}{\mu+\ell^2} \simeq \frac{1}{\sqrt{\mu}} \int_{L/\sqrt{\mu}}^\infty \frac{dx}{1+x^2} = O(\mu^{-1/2}),$$

completing the verification.

As for

$$\sup_{0\le x\le 1} E^*_\mu[p^2_h(x)] = \sup_{0\le x\le 1} \Big\{\sum_{\ell=5}^\infty \frac{(\phi^\mu_\ell(x))^2}{\lambda^\mu_\ell - 2\mu}\Big\},$$

the result will follow granted a uniform bound on $||\phi^\mu_\ell||_\infty$ independent of μ and of the index $\ell \ge 5$. (The intuition is thus: $\ell \ge 4$ corresponds to continuous spectrum for $\mu\to\infty$, and so the corresponding eigenfunctions should remain "flat" for large values of μ.) Once more, the ingredients of the verification are classical. It is convenient to consider the unscaled equation which reads $u''(x) + (6k^2\mathrm{cn}^2(x) + \gamma_\ell)u(x) = 0$ ($u(0) = u(\sqrt{\mu})$) with $\gamma_\ell > 0$ and of order $\ell^2/\sqrt{\mu}$ (note again 5.6). Then, for any ℓ up to

order $\sqrt{\mu}$, a comparison argument with the explicit $\ell = 4$ case ($\gamma \simeq 0$) will produce the desired bound. On the other hand, when $\ell > C\sqrt{\mu}$ with C large, well known arguments (see [11] for a model) will show that the solution is uniformly approximated (up to errors of $O(\ell^{-1})$) by a single trigonometric function with ℓ oscillations. The proof is complete.

Proof of Lemma 5.1 We concern ourselves with the upper bound, the lower bound being a simple consequence of Jensen's inequality. First note that

$$E_\mu^*\Big[e^{\theta\, F_0(p,\mu)},\, \|p\|_\infty \le \varepsilon\sqrt{\mu}\Big] \le E\Big[\exp\Big\{6\,\theta\,\Big(\int_0^1 p_\mu a_\mu^2 b_\mu\Big)\,\mathfrak{g}_0^2\mathfrak{g}_2 - \frac{\theta}{2}\Big(\int_0^1 a_\mu^4\Big)\,\mathfrak{g}_0^4\Big\}\Big].$$

That is, we only really use the quartic in the lowest mode to control the bad cubic term. Next, the integral on the right hand side may be performed in the $\mathfrak{g}_2$ variable, and we will be satisfied to show that there exists a choice of $\theta > 1$ and a $\delta > 0$ such that

$$36\,\theta^2\Big(\int_0^1 p_\mu a_\mu^2(x) b_\mu(x)\,dx\Big)^2 - \theta\int_0^1 a_\mu^4(x)dx \le -\frac{1}{\sqrt{\mu}}\delta \tag{5.7}$$

for all large enough μ. Now

$$\Big(\int_0^1 p_\mu(x)a_\mu^2(x)b_\mu(x)dx\Big)^2 = \frac{1}{\sqrt{\mu}}\frac{1}{8^4}\frac{(\int_0^{\sqrt{\mu}}\mathrm{cn}^4(x)\mathrm{sn}^2(x)\mathrm{dn}(x)dx)^2}{3\int_0^{\sqrt{\mu}}\mathrm{sn}^2(x)\mathrm{dn}^2(x)dx}$$

and

$$\int_0^1 a_\mu^4(x)dx = \frac{1}{\sqrt{\mu}}\frac{1}{8^4}\int_0^{\sqrt{\mu}}\mathrm{cn}^8(x)dx$$

up to (unimportant) errors of order μ^{-1}. Further, up to errors exponentially small in $\sqrt{\mu}$, the integrals on the right of the last two displays may be replaced by integrals over the whole line of the corresponding hyperbolic-trigonometric functions. That is, the validity of the desired inequality (5.7) is equivalent to whether

$$12\int_{-\infty}^{\infty}\mathrm{sech}^5(x)\tanh^2(x)dx < \int_{-\infty}^{\infty}\mathrm{sech}^8(x)dx\int_{-\infty}^{\infty}\mathrm{sech}^2(x)\tanh^2(x)dx.$$

Both sides of the latter may be worked out to read $12(\pi/16)^2 < (2/3)(32/35)$ which is indeed true. The proof is finished.

Proof of Lemma 5.2 Consider the first line comprising $F_1(p,\mu)$, see (5.4). That is, the estimate is detailed for $\tilde{F}_1(p,\mu) \equiv 2|\int_0^1 p_\mu p_h^3| + 6|\int_0^1 p_\mu p_\ell p_h^2| + 6|\int_0^1 p_\mu p_\ell^2 p_h|$. These terms are in a sense the most difficult as they involve the additional factor of $\sqrt{\mu}$ through p_μ. At the end we comment on how to deal with the remaining terms in the full F_1.

For the present task, it suffices to prove the following two types of estimate. First,

$$E_\mu^*\Big[\Big|\int_0^1 p_\mu p_h^3\Big|^m\Big]\to 0,\;\; E_\mu^*\Big[\Big|\int_0^1 p_\mu p_\ell p_h^2\Big|^m\Big]\to 0,\;\;\text{ and }\; E_\mu^*\Big[\Big|\int_0^1 p_\mu p_\ell^2 p_h\Big|^m\Big]\to 0. \tag{5.8}$$

as $\mu\to\infty$ for all m large enough. Second, for whatever $C > 0$

$$\limsup_{\mu\to\infty} E_\mu^*\Big[\exp\Big\{C\sqrt{\mu}\int_0^1|p_h|^3\Big\}, \|p\|_\infty \le \varepsilon\sqrt{\mu}\Big] < \infty, \tag{5.9}$$

$$\limsup_{\mu\to\infty} E_\mu^*\Big[\exp\Big\{C\sqrt{\mu}\int_0^1|p_\ell|p_h^2\Big\},\, \|p\|_\infty \le \varepsilon\sqrt{\mu}\Big] < \infty,$$

$$\text{and }\limsup_{\mu\to\infty} E_\mu^*\Big[\exp\Big\{C|\int_0^1 p_\mu p_\ell^2 p_h|\Big\},\, \|p\|_\infty \le \varepsilon\sqrt{\mu}\Big] < \infty$$

when $\varepsilon > 0$ is chosen appropriately.

Starting with (5.8) and working left to right we first have

$$\begin{aligned} E_\mu^*\Big[\Big|\int_0^1 p_\mu(x)p_h^3(x)dx\Big|^m\Big] &\le \mu^{m/2}\int_0^1 E_\mu^*[|p_h|^{3m}(x)]dx \\ &= C_3\,\mu^{m/2}\int_0^1\Big(E_\mu^*[p_h^2(x)]\Big)^{3m/2}dx \le C_4\,\mu^{-m/4}. \end{aligned}$$

Here we have used Jensen's inequality, the fact that $p_h(x)$ is Gaussian, and Lemma 5.4. In a similar fashion

$$\begin{aligned} &E_\mu^*\Big[\Big(\int_0^1 |p_\mu|(x)|p_\ell|p_h^3(x)dx\Big)^m\Big] \\ &\le\; C_4\,\mu^{m/2}\Big\{E_\mu^*\Big[\Big(\int_0^1 |a_\mu|(x)p_h^2(x)dx\Big)^m + \Big(\int_0^1 |b_\mu|(x)p_h^2(x)dx\Big)^m + \Big(\int_0^1 |c_\mu|(x)p_h^2(x)dx\Big)^m\Big\}. \end{aligned}$$

Restricting attention to the first term as a_μ has less decay compared with b_μ or c_μ we find that by Jensen's inequality and Lemma 5.4,

$$\mu^{m/2}\,E_\mu^*\Big[\Big(\int_0^1 |a_\mu|(x)p_h^2(x)dx\Big)^m \le C_5\mu^{m/2}\Big(\int_0^1 |a_\mu|^m(x)dx\Big)\Big(\sup_{0\le x\le 1} E_\mu^*[p_h(x)]\Big)^{m/2} \le C_6\,\mu^{-1/2}.$$

As for the last expectation in (5.8), it may be bounded by

$$\mu^{m/2}E_\mu^*\Big[\Big(\int_0^1 p_\ell^2(x)|p_h(x)|\,dx\Big)^m\Big] \le C_7\,\mu^{m/2}\Big\{E_\mu^*\Big[\Big(\int_0^1 a_\mu^2(x)|p_h(x)|\,dx\Big)^m\Big] + \text{ like terms in } b_\mu^2 \text{ and } c_\mu^2\Big\}.$$

Spelling out the first term involving we find

$$\begin{aligned} &\mu^{m/2}\,E_\mu^*\Big[\Big(\int_0^1 a_\mu^2(x)|p_h(x)|\,dx\Big)^m\Big] \\ &\le\; C_8\,\mu^{m/2}\Big(\int_0^1 |a_\mu|^{\frac{2m}{m-1}}(x)dx\Big)^{m-1}\int_0^1\Big(E_\mu^*[p_h^2(x)]\Big)^{m/2}dx \le C_9\mu^{-\frac{(m-1)}{2}} \end{aligned}$$

after an application of Hölder's inequality, and, again, the $\mu^{-1/2}$ decay of the moments of a_μ along with Lemma 5.4. Terms two and three are handled in the same way.

Turning now to exponential bounds (5.9), we require two additional observations. The first is that there exists an $M \ge 1$ so that

$$||p||_\infty \le \varepsilon\sqrt{\mu} \quad \text{implies} \quad ||p_\ell||_\infty \le M\varepsilon\sqrt{\mu}; \;\text{ and }\; ||p_h||_\infty \le M\varepsilon\sqrt{\mu}. \tag{5.10}$$

for some $M > 0$. To see this, first multiply the inequalities $-\varepsilon\sqrt{\mu} \le p(x) \le -\varepsilon\sqrt{\mu}$ through by $\phi_0^\mu(x) > 0$ to find that

$$|\mathfrak{g}_0|\int_0^1 a_\mu(x)\phi_0^\mu(x)dx \le \varepsilon\sqrt{\mu}\int_0^1 \phi_0^\mu(x)dx.$$

Next it may be checked that $\int_0^1 a_\mu\phi_0^\mu$ and $\int_0^1 \phi_0^\mu$ are of the same order, and so

$$|\mathfrak{g}_0| \le C_{10}\,\varepsilon\mu^{1/2}. \tag{5.11}$$

The verification of (5.10) is completed by showing that

$$|\mathfrak{g}_2| \le C_{11}\,\varepsilon\mu^{3/4} \quad \text{and} \quad |\mathfrak{g}_3| \le C_{12}\,\varepsilon\mu^{3/4} \tag{5.12}$$

on $||p||_\infty \le \varepsilon\sqrt{\mu}$. This is similar: the inequality is now multiplied through by $(||\phi_2^\mu||_\infty + \phi_2^\mu)$ (or $(||\phi_3^\mu||_\infty + \phi_3^\mu)$) and integrated. Since the path is mean zero this produces an inequality in $|\mathfrak{g}_2|$ (or $|\mathfrak{g}_3|$) alone which is equivalent to (5.12). The second fact we will need is the following upper bound on the sup-norm deviations of p_h. For all N large, there exist (positive) constants a and b independent of μ such that

$$P_\mu^*\Big(||p_h||_\infty > N\Big) \le a\,\exp\Big[-b\sqrt{\mu}N^2\Big]. \tag{5.13}$$

This is an immediate consequence of Borell's inequality (see [1]) and Lemma 5.4.

With those points established, we turn to the first expectation in question:

$$\begin{aligned}
&E_\mu^*\Big[\exp\Big\{C\sqrt{\mu}\int_0^1 |p_h|^3\Big\},\, ||p||_\infty \le \varepsilon\sqrt{\mu}\Big] \qquad (5.14)\\
&\le\; E_\mu^*\Big[\exp\Big\{C\sqrt{\mu}N\int_0^1 p_h^2\Big\},\, ||p_h||_\infty \le N\Big] + E_\mu^*\Big[\exp\Big\{C\mu\varepsilon\int_0^1 p_h^2\Big\},\, ||p_h||_\infty \ge N\Big]\\
&\le\; E_\mu^*\Big[\exp\Big\{C\sqrt{\mu}N\int_0^1 p_h^2\Big\}\Big] + \Big(E_\mu^*\Big[\exp\Big\{2C\mu\varepsilon\int_0^1 p_h^2\Big\}\Big]\Big)^{1/2}\Big(P(||p_h||_\infty \ge N)\Big)^{1/2}\\
&\le\; \exp\Big[CN\cdot\sum_{\ell\ge5}\frac{\sqrt{\mu}}{\lambda_\ell^\mu - 2\mu}\Big] + a\exp\Big[C\varepsilon\cdot\sum_{\ell\ge5}\frac{\mu}{\lambda_\ell^\mu-2\mu}\Big]\,\exp\Big[-\frac{1}{2}b\sqrt{\mu}N^2\Big].
\end{aligned}$$

Note that this last inequality holds as soon as μ is large enough and ε small enough so that both $CN\cdot(\sqrt{\mu}/(\lambda_2^\mu - 2\mu)) \simeq (1/3)CN\mu^{-1/2}$ and $2C\varepsilon\cdot(\sqrt{\mu}/(\lambda_2^\mu-2\mu)) \simeq (2/3)C\varepsilon$ are less than one-half. Now we recall that the sum $\sqrt{\mu}\sum_{\ell>2}\lambda_\ell^\mu$ is bounded by a fixed constant for $\mu\uparrow\infty$. Indeed, this is really the content of Lemma 5.4, this sum being finite along with $\limsup_{\mu\to\infty}\sum\frac{\sqrt{\mu}}{\mu+k^2} < \infty$. So the first term in the last line above is bounded independent of N, and the second will go to zero as $\mu\to\infty$ by appropriate choice of N.

The next integral is approached the same way. We find that

$$\begin{aligned}
&E_\mu^*\Big[\exp\Big\{C\sqrt{\mu}\int_0^1 |p_\ell|p_h^2\Big\},\, ||p||_\infty \le \varepsilon\sqrt{\mu}\Big]\\
&\le\; E_\mu^*\Big[\exp\Big\{C\sqrt{\mu}N^2\Big(||a_\mu||_1|\mathfrak{g}_0| + ||b_\mu||_1|\mathfrak{g}_2| + ||c_\mu||_1|\mathfrak{g}_3|\Big)\Big\}\Big]\\
&\quad + E_\mu^*\Big[\exp\Big\{C\cdot M\varepsilon\mu\int_0^1 p_h^2\Big\},\, ||p_h||_\infty \ge N\Big].
\end{aligned}$$

In the first term on the right hand side we have used $||p_h||_\infty \le N$ by explicitly restricting to that set of paths. Since $||a_\mu||_1 = O(\mu^{-1/2})$ and $||b_\mu||_1, ||c_\mu||_1 = O(\mu^{-3/4})$, this term is plainly bounded. In the second term we have used (5.10): $|p_\ell| \le M\varepsilon\sqrt{\mu}$ due the overall control of $||p||_\infty$. From this point this term yields to considerations identical to those for the second term in line of (5.14).

Last we come to bound the expectation involving a large constant multiple of

$$\begin{aligned}
&\Big|\int_0^1 p_\mu(x)p_\ell^2(x)p_h(x)dx\Big| \le \Big|\int_0^1 p_\mu(x)a_\mu^2(x)p_h(x)dx\Big|\,\mathfrak{g}_0^2 \qquad (5.15)\\
&\quad + \Big|\int_0^1 p_\mu(x)b_\mu^2(x)p_h(x)dx\Big|\,\mathfrak{g}_2^2 + 2\Big|\int_0^1 p_\mu(x)a_\mu(x)b_\mu(x)p_h(x)dx\Big|\,|\mathfrak{g}_0\mathfrak{g}_2| + etc;
\end{aligned}$$

the *etc* indicating like terms in $\mathfrak{g}_3^2$, $|\mathfrak{g}_0\mathfrak{g}_3|$ and $|\mathfrak{g}_2\mathfrak{g}_3|$. Making use of (5.11) and (5.12) we note that, for example,

$$\Big|\int_0^1 p_\mu(x)a_\mu(x)b_\mu(x)p_h(x)dx\Big|\,|\mathfrak{g}_0\mathfrak{g}_2| \le C_{13}\,\varepsilon\mu\,\Big|\int_0^1 \mathrm{sn}(\sqrt{\mu}x)a_\mu(x)\tilde{\phi}_2^\mu(x)p_h(x)dx\Big|\,|\mathfrak{g}_0|.$$

Similarly, each term on the right hand side of (5.15) may be bounded in turn by (constant multiples of) expressions of the form

$$\mu\varepsilon\Big|\int_0^1 \psi(\sqrt{\mu}x)p_h(x)dx\Big||\mathfrak{g}_\ell| \equiv \mu\varepsilon|\mathcal{G}(\psi)||\mathfrak{g}_\ell|$$

with $\ell = 0, 2$, or 3 and ψ a smooth periodic function for which $\limsup_{\mu\to\infty}\int_0^{\sqrt{\mu}}|\psi(x)|dx < \infty$. While ψ differs in each appearance it now suffices to show that for any such ψ, ε may be chosen small enough so that

$$E\Big[\exp\Big\{\mu\varepsilon|\mathcal{G}(\phi)||\mathfrak{g}_0|\Big\}\Big] \le C_{14}E\Big[\exp\Big\{\mu^2\varepsilon^2\mathcal{G}^2(\phi)\Big\}\Big] \le C_{15}$$

independently of μ. Now since $\mathcal{G}(\phi)$ is itself a mean zero Gaussian random variable, this last display is true as long as

$$\mu^2E[\mathcal{G}^2(\psi)] = \mu^2\sum_{\ell\ge 5}\frac{1}{\lambda_\ell^\mu - 2\mu}\Big(\int_0^1 \psi(\sqrt{\mu}x)\phi_\ell^\mu(x)dx\Big)^2 \tag{5.16}$$

may be bounded independently of μ. In this direction, we have the estimate

$$\int_0^1 \psi(\sqrt{\mu}x)\phi_\ell^\mu(x)dx \le C_{15}\Big(\frac{1}{\sqrt{\mu}}\wedge\frac{1}{\ell}\Big) \tag{5.17}$$

which is explained as follows. In Lemma 5.4 it is remarked that there is a uniform L^∞ bound on the high eigenfunctions $|\phi_\ell^\mu|$. The $\mu^{-1/2}$ decay may then always be extracted from the same decay of ψ in L^1. On the other hand, as also remarked in Lemma 5.4, for high values of the index ϕ_ℓ^μ is well approximated by a trigonometric function of ℓ-turns. The ℓ^{-1} factor then follows from the lemma of Riemann-Lebesgue. Finally, substituting (5.17) into (5.16) produces

$$\mu^2E[\mathcal{G}^2(\psi)] \le C_{16}\Big(\mu^2\sum_{4\le\ell\le\sqrt{\mu}}\frac{1}{\mu+\ell^2}\cdot\frac{1}{\mu} + \mu^2\sum_{\sqrt{\mu}\le\ell<\infty}\frac{1}{\mu+\ell^2}\cdot\frac{1}{\ell^2}\Big),$$

and it is readily checked that the latter remains bounded as $\mu\to\infty$.

We close with some comments regarding the remaining terms in $F_1(p,\mu)$. Looking at line two of (5.4) it is plain at this point that the linear term $|\int_x^y p| \le \int_0^1 |p|$ poses no difficulty. On the other hand $\int_0^1 |p_\ell||p_h|^3 \le M\varepsilon\sqrt{\mu}\int_0^1 |p_h|^3$ on the domain of integration which reduces the term to a type already dealt with. Also, expanding the low modes out in $|\int_0^1 p_\ell^3 p_h|$ and employing (5.11) and (5.12) make this term comparable to $|\int_0^1 p_\mu p_\ell^2 p_h|$. Finally, the last line in (5.4) is explicit in terms of the low modes. It may be done by hand, cancellations in the cross terms (as in (5.2)) being of great importance. The tedious details are not reported. The proof is complete.

Proof of Lemma 5.3 We actually prove that, on the set of paths $\{p : ||p||_\infty \le \varepsilon\sqrt{\mu}\}$, there is a fixed constant so that

$$\Big|R(p_\mu + p) - R(p_\mu)\Big| \le C_0\,R(p_\mu)\,\frac{1}{\sqrt{\mu}}\int_0^1 |p|.$$

The L^θ estimate for $\theta > 1$ then follows from Jensen's Inequality and the first conclusion of Lemma 5.4.

Introduce the un-normalized extremum $p^*(x) = k\,\mathrm{sn}(\sqrt{\mu}x, k)$ and the functional

$$\mathcal{R}(a,p) = \int_0^1 \tilde{\phi}_1^\mu(x+a)p(x)dx = \int_0^1 \mathrm{cndn}(\sqrt{\mu}(x+a))p(x)dx.$$

It is the derivative of $\mathcal{R}$ in a which figures in the definition of R. Clearly, for any path p, $\mathcal{R}(a,p)$ is is an analytic function of a, and $\mathcal{R}'(a,p) \le C||p||_\infty$ with a constant independent of a. Further, at the

extremum p^*, $\mathcal{R}(a,p^*)$ has exactly two zeros at $a=0$ and $a=1/2$, and the derivatives at those points satisfy $\mathcal{R}'(0,p^*) = -\mathcal{R}'(1/2,p^*)$. Also, one may check that $|\mathcal{R}'(0,p^*)|$ converges to a positive constant as $\mu\uparrow\infty$. It follows that if p satisfies $||p||_\infty \le \sqrt{\mu}\varepsilon$, then, for all large μ, $\mathcal{R}(a, p^*+p/\sqrt{\mu})$ has exactly two zeros, $r_1 = r_1(p)$ and $r_2 = r_2(p)$, lying within neighborhoods of radius ε from 0 and 1/2 respectively. In fact, $|r_1|$ and $|r_2 - 1/2|$ may be bounded in a more useful way by a constant multiple of $\frac{1}{\sqrt{\mu}}\int_0^1 |p|$. Take the case of r_1:

$$\begin{aligned}\frac{1}{\sqrt{\mu}}\int_0^1 |p(x)|dx &\geq \frac{1}{\sqrt{\mu}}\Big|\int_0^1 \tilde{\phi}_1^\mu(x+r_1)p(x)\,dx\Big| \\ &= \Big|\int_0^1 \Big(\tilde{\phi}_1^\mu(x) - \tilde{\phi}_1^\mu(x+r_1)\Big)p^*(x)\,dx\Big| = |r_1(p)|\,\Big|\mathcal{R}'(\tilde{r},p^*)\Big|\end{aligned} \tag{5.18}$$

with some $\tilde{r}$ between 0 and r_1, that is, $|\tilde{r}| = O(\varepsilon)$. Since there exists a $\delta>0$ with $|\mathcal{R}'(0,p^*)|>\delta$ for all large μ, by analyticity $|\mathcal{R}'(\tilde{r},p^*)|$ satisfies a similar lower bound. The claimed bound on r_1 follows.

Next let us write $\mathcal{R}_0 = |\mathcal{R}'(0,p^*)|$, $\mathcal{R}_1 = |\mathcal{R}'(r_1, p^*+p/\sqrt{\mu})|$, and $\mathcal{R}_2 = |\mathcal{R}'(r_2,p^*+p/\sqrt{\mu})|$ in terms of which

$$\Big|R(p_\mu+p)-R(p_\mu)\Big| = \frac{\sqrt{\mu}}{\sqrt{\int_0^1(\tilde{\phi}_1^\mu)^2}}\,\Big|\frac{\mathcal{R}_0}{2} - \frac{\mathcal{R}_1\mathcal{R}_2}{\mathcal{R}_1+\mathcal{R}_2}\Big| \le \frac{\sqrt{\mu}}{\sqrt{\int_0^1(\tilde{\phi}_1^\mu)^2}}\,\left\{\Big|\frac{\mathcal{R}_1(\mathcal{R}_0-\mathcal{R}_2)}{\mathcal{R}_1+\mathcal{R}_2}\Big| + \Big|\frac{\mathcal{R}_2(\mathcal{R}_0-\mathcal{R}_1)}{\mathcal{R}_1+\mathcal{R}_2}\Big|\right\}.$$

It follows that it enough to show that $|\mathcal{R}_0 - \mathcal{R}_1|$ and $|\mathcal{R}_0-\mathcal{R}_1|$ are bounded by $C_1\mathcal{R}_0\mu^{-1/2}\int_0^1|p|$ or, what is the same, by $C_2\int_0^1|p|$. (That $\mathcal{R}_{1,2}/(\mathcal{R}_1+\mathcal{R}_2) = O(1)$ is plain.) Consider the difference of $\mathcal{R}_0$ and $\mathcal{R}_1$, the other estimate being identical. On the set $\{p : ||p||_\infty \le \sqrt{\mu}\,\varepsilon\}$ we have

$$\begin{aligned}&|\mathcal{R}_0 - \mathcal{R}_1| \\ &= \Big|\int_0^1 (\tilde{\phi}_1^\mu)'(x)p^*(x)dx - \int_0^1 (\tilde{\phi}_1^\mu)'(x+r_1)\Big(p^*(x) + \frac{1}{\sqrt{\mu}}p(x)\Big)dx\Big| \\ &\le \int_0^1 \Big|(\tilde{\phi}_1^\mu)'(x) - (\tilde{\phi}_1^\mu)'(x+r_1)\Big||p^*(x)|\,dx + \frac{1}{\sqrt{\mu}}\int_0^1\Big|(\tilde{\phi}_1^\mu)'(x+r_1)\Big||p(x)|dx \\ &\le \mu\,|r_1(p)|\int_0^1\Big|(\mathrm{cndn})''(\sqrt{\mu}(x+r^*))\,\mathrm{sn}(\sqrt{\mu}x)\Big|\,dx + \int_0^1 |(\mathrm{cndn})'(\sqrt{m}(x+r_1))|\,|p(x)|\,dx\end{aligned}$$

for some r^*, $-|r_1|\le r^* \le |r_1|$. In line two we have supposed that ε is small enough that the absolute values in the definition of $\mathcal{R}_0$ and $\mathcal{R}_1$ may be left off. To finish, note that $|(\mathrm{cndn})'(\cdot)|\le 2$,

$$\int_0^1 \Big|(\mathrm{cndn})''(\sqrt{\mu}(x+r^*))\,\mathrm{sn}(\sqrt{\mu}x)\Big|\,dx = O(\mu^{-1/2})$$

if r^* is bounded with probability one, and last, from (5.18), $|r_1(p)| \le C_3\mu^{-1/2}\int_0^1|p|$. The proof is complete.

6 The Gaussian correction

All would be for naught if we could not compute the "Gaussian correction":

$$Z(\mu) = E_0^0\Big[\exp\{-\frac{1}{2}\int_0^1(q_\mu(x)-2\mu)p^2(x)dx\}\Big]P_0\Big(\int_0^1\phi_1^\mu(x)p(x)dx = 0\Big) \tag{6.1}$$

$(= Z^*P_0(\int_0^1\phi_1^\mu p = 0))$. The properties of the operator $Q_\mu = -d^2/dx^2 + q_\mu$ outlined in Section 4 now play an essential role. As in the proof of Lemma 4.1, it is convenient to renormalize and split the E_0^0

integral into two pieces as follows

$$E_0^0\Big[\exp\Big\{-\frac{1}{2}\int_0^1 (q_\mu - 2\mu)p^2\Big\}\Big] = E_{q_\mu}\Big[\exp\Big\{\mu\int_0^1 p^2\Big\}\ \Big|\ \int_0^1 p = 0, \int_0^1 \phi_1^\mu p = 0\Big] \times E_0^0\Big[\exp\Big\{-\frac{1}{2}\int_0^1 q_\mu p^2\Big\}\Big]. \tag{6.2}$$

Again, E_{q_μ} denotes the mean-value with respect to the Gaussian weight $Z_{q_\mu}^{-1}\exp\left[-1/2\int_0^1 q_\mu p^2\right] \times dCBM$. We recall that the point of this splitting is that, without any conditioning, the latter is a proper probability measure on periodic paths (*i.e.*, $Z_{q_\mu} < \infty$).

Taking advantage of this, the first integral in (6.2) may be evaluated as follows.

Proposition 6.1 *For all $\mu > 0$,*

$$\begin{aligned} & E_{q_\mu}\Big[\exp\Big\{\mu\int_0^1 p^2\Big\}\ \Big|\ \int_0^1 p = 0, \int_0^1 \phi_1^\mu p = 0\Big] \\ = & \Big[\Big(1-\frac{2\mu}{\lambda_0^\mu}\Big)\Big(1-\frac{2\mu}{\lambda_1^\mu}\Big)\Big(1-\frac{2\mu}{\lambda_4^\mu}\Big)\Big]^{1/2} \times \Big[\frac{\lambda_4^\mu c_0^2 + \lambda_0^\mu c_4^2}{(\lambda_4^\mu - 2\mu)c_0^2 + (\lambda_0^\mu - 2\mu)c_4^2}\Big]^{1/2} \times \sqrt{\frac{\Delta^2(0)-4}{\Delta^2(2\mu)-4}}. \end{aligned}$$

Here $\Delta(\lambda)$ is the discriminant of Q_μ and the constants c_0 and c_4 are as defined in (4.4).

The computation behind the second piece is more involved. The result is:

Proposition 6.2 *There is the explicit formula*

$$E_0^0\Big[\exp\Big\{-\frac{1}{2}\int_0^1 q_\mu p^2\Big\}\Big] P_0\Big(\int_0^1 \phi_1^\mu p = 0\Big) = \Big[\frac{2\pi}{\lambda_1^\mu}\Big(\frac{c_0^2}{\lambda_0^\mu} + \frac{c_4^2}{\lambda_4^\mu}\Big)\Big]^{-1/2} \times \frac{1}{\sqrt{\Delta^2(0)-4}}; \tag{6.3}$$

the notation being the same as in the previous result.

Note what has been accomplished: by Propositions 6.1 and 6.2 and Hochstadt's formula (4.1), $Z(\mu)$ is now expressed completely in terms of the simple spectrum of Q_μ which we know explicitly.

Proof of Proposition 6.1 Once more we bring in the expansion of the path $p(x) = \sum_{\ell=0}^\infty \frac{1}{\sqrt{\lambda_\ell^\mu}}\phi_\ell^\mu(x)\mathfrak{g}_\ell$ under P_{q_μ}. This translates the expectation of interest to: with E the mean corresponding to the independent Gaussian $\mathfrak{g}$'s,

$$\begin{aligned} & E_{q_\mu}\Big[\exp\Big\{\mu\int_0^1 p^2(x)dx\Big\}\ \Big|\ \int_0^1 p(x)dx = 0, \int_0^1 \phi_1^\mu(x)p(x)dx = 0\Big] \\ = & E\Big[\exp\Big\{\sum_{\ell=0}^\infty \frac{\mu}{\lambda_\ell^\mu}\mathfrak{g}_\ell^2\Big\}\ \Big|\ \mathfrak{g}_1 = 0, \frac{c_0}{\sqrt{\lambda_0^\mu}}\mathfrak{g}_0 - \frac{c_4}{\sqrt{\lambda_4^\mu}}\mathfrak{g}_4 = 0\Big] \\ = & \Big[\prod_{2\le\ell\le\infty,\ell\ne4}\Big(1-\frac{2\mu}{\lambda_\ell^\mu}\Big)\Big]^{-1/2} E\Big[\exp\Big\{\frac{\mu}{\lambda_0^\mu}\mathfrak{g}_0^2 + \frac{\mu}{\lambda_0^\mu}\mathfrak{g}_4^2\Big\}\Big|\frac{c_0}{\sqrt{\lambda_0^\mu}}\mathfrak{g}_0 - \frac{c_4}{\sqrt{\lambda_4^\mu}}\mathfrak{g}_4 = 0\Big]. \end{aligned} \tag{6.4}$$

The integral in the last line already appeared in the proof of Lemma 4.1; we now detail its evaluation:

$$\begin{aligned} & E\Big[\exp\Big\{\frac{\mu}{\lambda_0^\mu}\mathfrak{g}_0^2 + \frac{\mu}{\lambda_4^\mu}\mathfrak{g}_4^2\Big\}\Big|\frac{c_0}{\sqrt{\lambda_0^\mu}}\mathfrak{g}_0 - \frac{c_4}{\sqrt{\lambda_4^\mu}}\mathfrak{g}_4 = 0\Big] \\ = & \sqrt{2\pi\Big(\frac{c_0^2}{\lambda_0^\mu} + \frac{c_4^2}{\lambda_4^\mu}\Big)} \times \int_{-\infty}^\infty \exp\Big\{-\frac{1}{2}\Big[(1-2\mu/\lambda_0^\mu)\frac{c_4^2}{\lambda_4^\mu} + (1-2\mu/\lambda_4)\frac{c_0^2}{\lambda_0^\mu}\Big]x^2\Big\}\frac{dx}{2\pi} \\ = & \Big(\frac{\lambda_4^\mu c_0^2 + \lambda_0^\mu c_4^2}{(\lambda_4^\mu - 2\mu)c_0^2 + (\lambda_0^\mu - 2\mu)c_4^2}\Big)^{1/2}. \end{aligned}$$

Concerning the prefactor in (6.4), when restored to a product over the full range, one over its square reads as $P(\lambda) \equiv \prod_{\ell=0}^{\infty}(1-\lambda/\lambda_\ell^\mu)$ evaluated at $\lambda = 2\mu$. The behavior of λ_ℓ^μ for $\ell\uparrow\infty$ shows that the latter is entire function of order $1/2$. Also, as $P(\lambda)$ vanishes only at the periodic spectrum of Q_μ, one concludes that it is a constant multiple of $\Delta^2(\lambda)-4$. In other words,

$$\Big[\prod_{2\leq\ell\leq\infty,\ell\neq 4}\Big(1-\frac{2\mu}{\lambda_\ell^\mu}\Big)\Big]^{-1/2} = \sqrt{\frac{\Delta^2(0)-4}{\Delta^2(2\mu)-4}} \times \Big[\Big(1-\frac{2\mu}{\lambda_0^\mu}\Big)\Big(1-\frac{2\mu}{\lambda_1^\mu}\Big)\Big(1-\frac{2\mu}{\lambda_4^\mu}\Big)\Big]^{1/2}.$$

The proof is finished.

Proof of Proposition 6.2 . The goal is to convert the CBM integral to one over Brownian bridge paths. First though, the conditionings $\int_0^1 p = 0$ and $\int_0^1 p\phi_1^\mu$ are removed as follows

$$\begin{aligned} & E_0^0\Big[\exp\Big\{-\frac{1}{2}\int_0^1 q_\mu(x)p^2(x)dx\Big\}\Big] \qquad (6.5)\\ = \;& P_0^{-1}\Big(\int_0^1 \phi_1^\mu p = 0\Big) P_{q_\mu}\Big(\int_0^1 p = 0, \int_0^1 \phi_1 p = 0\Big) \times CBM\Big[e^{-\frac{1}{2}\int_0^1 q_\mu(x)p^2(x)dx}\Big]. \end{aligned}$$

The first factor, while computable, cancels the same object in the numerator of (6.3). The second is of a similar nature: under P_{q_μ}, the variables $\int_0^1 p(x)dx$ and $\int_0^1 \phi_1(x)p(x)dx$ are independent Gaussians:

$$\begin{aligned} P_{q_\mu}\Big(\int_0^1 p = 0, \int_0^1 \phi_1^\mu p = 0\Big) &= P\Big(\frac{c_0}{\sqrt{\lambda_0^\mu}}\mathfrak{g}_0 - \frac{c_4}{\sqrt{\lambda_4^\mu}}\mathfrak{g}_4 = 0\Big) P\Big(\frac{1}{\sqrt{\lambda_1^\mu}}\mathfrak{g}_1 = 0\Big) \qquad (6.6)\\ &= \frac{1}{\sqrt{2\pi(c_0^2/\lambda_0^\mu + c_4^2/\lambda_4^\mu)}} \times \frac{1}{\sqrt{2\pi/\lambda_1^\mu}}, \end{aligned}$$

as advertised. Turning to the third factor in (6.5), we now unravel the periodic boundary conditions. From the definition,

$$\begin{aligned} CBM\Big[\exp\Big\{-\frac{1}{2}\int_0^1 q_\mu(x)p^2(x)dx\Big\}\Big] &= \int_{-\infty}^{\infty} BM_{00}\Big[\exp\Big\{-\frac{1}{2}\int_0^1 q_\mu(x)(p(x)+c)^2dx\Big\}\Big]\frac{dc}{\sqrt{2\pi}} \qquad (6.7)\\ &= Z_{q_\mu,0}\int_{-\infty}^{\infty}\exp\Big\{-\frac{1}{2}c^2\int_0^1 q_\mu(x)dx\Big\}E_{q_\mu,0}\Big[\exp\Big\{-c\int_0^1 q_\mu p\Big\}\Big]\frac{dc}{\sqrt{2\pi}}. \end{aligned}$$

Here, in line two, we have renormalized yet again to introduce the measure $P_{q_\mu,0}$ on tied paths with weight $Z_{q_\mu,0} = BM_{00}[\exp\{-(1/2)\int_0^1 q_\mu p^2\}]$. Said differently, $P_{q_\mu,0}$ is the Gaussian measure with inverse covariance operator $Q_{\mu,0} = -d^2/dx^2 + q_\mu$ over paths $p(x)$ subject to $p(0) = p(1) = 0$. As such, the integrand in (6.7) may be worked out as

$$\exp\Big\{-\frac{1}{2}c^2\int_0^1 q_\mu(x)dx\Big\}E_{q_\mu,0}\Big[\exp\Big\{-c\int_0^1 q_\mu(x)p(x)dx\Big\}\Big] = \exp\Big\{-\frac{1}{2}c^2\Big[\int_0^1 q_\mu(x)\Big(1-(Q_{\mu,0}^{-1}q_\mu)(x)\Big)dx\Big]\Big\}$$

by simply using the definition of the Green's function $Q_{\mu,0}^{-1}$. Next, with $\psi(x) = (\mathfrak{G}_0^{-1}q_\mu)(x)$ it is immediate that

$$\int_0^1 q_\mu(x)\Big(1-(\mathfrak{G}_0^{-1}q_\mu)(x)\Big)dx = -\int_0^1 \psi''(x)dx \equiv \int_0^1\Big[\psi_0''(x)+\psi_1''(x)\Big]dx$$

in which $\psi_0(x)$ and $\psi_1(x)$ are the increasing/decreasing solutions of $\psi''(x) = q_\mu(x)\psi(x)$ over $0 < x < 1$ subject to $\psi_0(0) = 0$, $\psi_0(1) = 1$ and $\psi_1(0) = 1$, $\psi_1(1) = 0$. In terms of the normalized cosine and sine-like solutions at $\lambda = 0$, we have $\psi_0(x) = y_1(x,0) - (y_1(1,0)/y_2(1,0))y_2(x,0)$, $\psi_1(x) = y_2(x,0)/y_2(1,0)$, and so

$$\int_0^1\Big[\psi_0''(x)+\psi_1''(x)\Big]dx = y_1'(1,0) - \frac{y_1(1,0)}{y_2(1,0)}\Big[y_2'(1,0)-1\Big] + \frac{1}{y_2(1,0)}\Big[y_2'(1,0)-1\Big]. \qquad (6.8)$$

Finally, we bring in the classical computation (see [23]),

$$Z_{q_\mu 0} = BM_{00}\Big[\exp\Big\{-\frac{1}{2}\int_0^1 q_\mu(x)p^2(x)dx\Big\}\Big] = \frac{1}{\sqrt{y_2(1,0)}},$$

which, when combined with (6.7) through (6.8), gives

$$\begin{aligned} CBM\Big[\exp\Big\{-\frac{1}{2}\int_0^1 q_\mu(x)p^2(x)dx\Big\}\Big] &= \Big[y_1(1,0)+y_2'(1,0)-2\Big]^{-1/2} \\ &= \Big[\Big(y_1(1/2,0)+y_2'(1/2,0)\Big)^2-4\Big]^{-1/2} \equiv \frac{1}{\sqrt{\Delta^2(0)-4}}. \end{aligned} \tag{6.9}$$

Here we have made use of the Wronskian identity $1 = y_1y_2' - y_1'y_2$ in line one and the connection formula $y_{1,2}(1) = y_{1,2}(1/2)y_1(1/2) + y_{1,2}'(1/2)y_2(1/2)$ in line two. The proof is finished.

7 Putting it all together: final asymptotics

The results through this point are summarized in the following Theorem.

Theorem 7.1 *For large $\mu > 0$,*

$$f(\mu) = \sqrt{\frac{2}{\pi}}A(p_\mu)R(p_\mu)Z(\mu)\exp\Big[I_\mu(p_\mu)\Big](1+o(1)) \tag{7.1}$$

with $A(\cdot), R(\cdot), I_\mu(\cdot)$ and $Z(\mu)$ as defined in (1.2), (3.4), (2.1) and (6.1) respectively. Everything on the right hand side is an explicit functional of $p_\mu = \sqrt{\mu}k sn(\cdot,k)$ and the simple spectrum of Q_μ.

This is really the main result of the paper. By working out of the asymptotics of the individual objects on its right hand side, (7.1) may be translated to the statement given in Theorem 1.1.

To begin, that $I_\mu(p_\mu) = -8/3\mu^{3/2}$ up to exponentially small corrections in $\sqrt{\mu}$ has already been noted. By similar considerations we find that,

$$\begin{aligned} R(p_\mu) &= \frac{1}{2}\mu k^2 \frac{\int_0^1 \mathrm{sn}^2(\sqrt{\mu}x)\Big(\mathrm{dn}^2(\sqrt{\mu}x)+k^2\mathrm{cn}^2(\sqrt{\mu}x)\Big)dx}{\sqrt{\int_0^1 \mathrm{cn}^2(\sqrt{\mu}x)\mathrm{dn}^2(\sqrt{\mu}x)dx}} \\ &= \sqrt{2}\mu^{3/4}\frac{\int_{-\infty}^{\infty}\tanh^2(x)\mathrm{sech}^2(x)dx}{\sqrt{\int_{-\infty}^{\infty}\mathrm{sech}^4(x)dx}}\Big(1+O\Big(e^{-\sqrt{\mu}/4}\Big)\Big) = \sqrt{\frac{2}{3}}\mu^{3/4}\Big(1+O\Big(e^{-\sqrt{\mu}/4}\Big)\Big). \end{aligned} \tag{7.2}$$

For $A(p_\mu) = A_+(p_\mu)A_-(p_\mu)$, we have first by direct computation:

$$\begin{aligned} A_+(p_\mu) = \int_0^1 e^{2k\sqrt{\mu}\int_0^x \mathrm{sn}(\sqrt{\mu}x')dx'}dx &= \frac{1}{\sqrt{(1-k)^2}}\Big(\int_0^1 \mathrm{dn}^2(\sqrt{\mu}x)+k^2\mathrm{cn}^2(\sqrt{\mu}x)\Big)dx \\ &= \frac{1}{8\sqrt{\mu}}e^{\sqrt{\mu}}\times\Big(1+O\Big(e^{-\sqrt{\mu}/4}\Big)\Big). \end{aligned}$$

The second half of A responds to

$$\begin{aligned} A_-(p_\mu) &= \int_0^1 e^{-2k\sqrt{\mu}\int_0^x \mathrm{sn}(\sqrt{\mu}x')dx'}dx = \frac{2}{\sqrt{\mu}}\int_0^{\sqrt{\mu}/4} e^{-2k\int_0^x \mathrm{sn}(x')dx'}dx + O\Big(e^{-\sqrt{\mu}}\Big) \\ &= \frac{2}{\sqrt{\mu}}\int_0^{\infty} e^{-2\int_0^x \tanh(x')dx'}dx\times\Big(1+O\Big(e^{-\sqrt{\mu}/4}\Big)\Big) = \frac{2}{\sqrt{\mu}}\times\Big(1+O\Big(e^{-\sqrt{\mu}/4}\Big)\Big), \end{aligned}$$

the integral in the second line being easily computed.

Last we turn to the asymptotics of $Z(\mu)$. Examining the results of Propositions 6.1 and 6.2, we see that we have two objects not involving Hill's discriminant:

$$\Big[\Big(1-\frac{2\mu}{\lambda_0^\mu}\Big)\Big(1-\frac{2\mu}{\lambda_1^\mu}\Big)\Big(1-\frac{2\mu}{\lambda_4^\mu}\Big)\Big(\frac{\lambda_4^\mu c_0^2+\lambda_0^\mu c_4^2}{(\lambda_4^\mu-2\mu)c_0^2+(\lambda_0^\mu-2\mu)c_4^2}\Big)\Big]^{1/2} = \frac{8}{3\sqrt{2}}\mu^{1/4}e^{-\sqrt{\mu}/2}\times\Big(1+O\Big(\frac{1}{\sqrt{\mu}}\Big)\Big),$$

and

$$\Big[\frac{2\pi}{\lambda_1^\mu}\Big(\frac{c_0^2}{\lambda_0^\mu}+\frac{c_4^2}{\lambda_4^\mu}\Big)\Big]^{-1/2} = \sqrt{\frac{6}{\pi}}\mu\times\Big(1+O\Big(\frac{1}{\sqrt{\mu}}\Big)\Big). \tag{7.3}$$

Here we have once again made use $1-k^2 = 1-k^2(\mu) = 16e^{-\sqrt{\mu}/2}(1+o(1))$, the list (4.2), as well as $c_0(\mu)=\sqrt{6}\mu^{-1/4}+O(e^{-\sqrt{\mu}/4})$ and $c_4(\mu)=1+O(e^{-\sqrt{\mu}/4})$, as pointed out in the proof of Lemma 4.1. We now make use of Hochstadt's formula to estimate the discriminant for large values of μ.

Proposition 7.1 *It holds*

$$\Big(\Delta^2(2\mu)-4\Big)^{-1/2} = e^{-\sqrt{\mu}}\times\Big(1+O\Big(\frac{1}{\sqrt{\mu}}\Big)\Big) \tag{7.4}$$

as $\mu\to\infty$.

Gathering one over the right hand side of (7.4) together with displays (7.2) through (7.3) produces the form of the result originally stated in Theorem 1.1.

Proof of Proposition 7.1 Starting from $\Delta(\cdot)=2\cos\psi(\cdot)$ with ψ defined in (4.1) we have that

$$\begin{aligned}\Delta(2\mu) &= 2\cos\left(\frac{\sqrt{-1}}{2}\int_{\lambda_0^\mu}^{2\mu}\frac{(s-(\lambda_1')^\mu)(s-(\lambda_2')^\mu)}{\sqrt{-(s-\lambda_0^\mu)\cdots(s-\lambda_4^\mu)}}ds\right) \\ &= -2\cosh\left(\frac{\sqrt{\mu}}{2}\int_{\lambda_1}^{2}(s-\lambda_1')(s-\lambda_2')\frac{ds}{\sqrt{R(s)}}\right).\end{aligned} \tag{7.5}$$

Besides an obvious change of variables, the second line makes use of the fact $\psi(\lambda_1^\mu)=\pi$. $R(s)$ is shorthand for $(s-\lambda_0)(s-\lambda_1)\times(\lambda_2-s)(\lambda_3-s)(\lambda_4-s)$ which is non-negative for $s\in[\lambda_1,2]$. Note that $\lambda_0,\dots,\lambda_4$, λ_1', and λ_2' now refer to the spectral points of the unscaled operator with periodic boundary conditions over $[0,2K=\sqrt{\mu}/2]$. While these points still depend on μ, we will show that

$$\int_{\lambda_1}^{2}(s-\lambda_1')(s-\lambda_2')\frac{ds}{\sqrt{R(s)}}\to 2$$

as $\mu\to\infty$. First though we must pin down the behavior of $\lambda_1'\in[\lambda_1,\lambda_2]$ and $\lambda_2'\in[\lambda_3,\lambda_4]$ in that limit.

Returning to Hochstadt's result we know that

$$0=\int_{\lambda_1}^{\lambda_2}(s-\lambda_1')(s-\lambda_2')\frac{ds}{\sqrt{R(s)}} \quad\text{and}\quad 0=\int_{\lambda_3}^{\lambda_4}(s-\lambda_1')(s-\lambda_2')\frac{ds}{\sqrt{R(s)}} \tag{7.6}$$

from which it may be immediately inferred that $|\lambda_1'-\lambda_1|$ and $|\lambda_2'-\lambda_3|$ tend to zero as $\mu\to\infty$. (Recall that, as $\mu\to\infty$, $\lambda_0\simeq\lambda_1=2+O(e^{-\sqrt{\mu}/2})$ and $\lambda_2\simeq\lambda_3=5+O(e^{-\sqrt{\mu}/2})$.) Indeed, if λ_2' were to remain greater than $\lambda_3+\delta$ for a fixed $\delta>0$, the second equality in (7.6) would require that $\lambda_1'\to 5$ as $\mu\to\infty$. However, with $\lambda_1'\simeq 5$ there would be nothing to balance the singularity at the lower limit of the integral in the first equality. Given now that $\lambda_2'\to 5$, it must be that $\lambda_1'\to 2$ in order that the first integral remains finite.

For sharper information it is convenient to introduce the (small) parameter $\varepsilon \equiv 1 - k^2$, whereupon

$$\lambda_0 = 2 - \varepsilon - 3/4\,\varepsilon^2 - O(\varepsilon^3), \qquad \lambda_1 = 2 - \varepsilon, \qquad\qquad (7.7)$$
$$\lambda_2 = 5 - 4\varepsilon, \qquad \lambda_3 = 5 - \varepsilon, \qquad \lambda_4 = 6 - 4\varepsilon - O(\varepsilon^2).$$

We further denote $\lambda_1' = \lambda_1 + \delta_1(\varepsilon)$ and $\lambda_2' = \lambda_3 + \delta_2(\varepsilon)$. The asymptotics of $\delta_2(\varepsilon)$ as $\varepsilon \to 0$ may now be determined from the second relation in (7.6). Note first that since $(s - \lambda_1')/\sqrt{(s-\lambda_0)(s-\lambda_1)} = 1 + o(1)$ for $s \in [\lambda_3, \lambda_4]$ and $\varepsilon \to 0$, it suffices to consider

$$0 = \int_{\lambda_3}^{\lambda_4} \frac{(s - \lambda_3 - \delta_2(\varepsilon))}{\sqrt{(s-\lambda_2)(s-\lambda_3)(\lambda_4 - s)}} ds$$

in order to gather the leading order behavior of $\delta_2(\varepsilon)$. Next, shifting variables and employing (7.7), this last equality is the same as

$$\int_0^1 \sqrt{\frac{s}{s+3\varepsilon}} \frac{ds}{\sqrt{1-s}} = \delta_2(\varepsilon) \int_0^1 \frac{ds}{\sqrt{s(s+3\varepsilon)(1-s)}}$$

us to negligible errors. From here one quickly concludes that $\delta_2(\varepsilon) = 2(\log|\varepsilon|)^{-1}(1 + o(1))$. Armed with this information, similar considerations brought to bear in the first relation in (7.6) lead to the estimate $\delta_1(\varepsilon) = 4(\log|\varepsilon|)^{-1}(1 + o(1))$.

Now we may return to the main integral at hand:

$$\begin{aligned}
\int_{\lambda_1}^2 (s - \lambda_1')(s-\lambda_2') \frac{ds}{\sqrt{R(s)}} &= \int_0^{\varepsilon} (s - \delta_1(\varepsilon))(s - (\lambda_2' - \lambda_1)) \frac{ds}{\sqrt{R(s+\lambda_1)}} \\
&= \Big(\int_0^1 \frac{(\delta_1(\varepsilon) - \varepsilon s)}{\sqrt{s(s + 3/4\varepsilon + o(\varepsilon))}} \frac{(3 + \delta_2(\varepsilon) - \varepsilon s) ds}{\sqrt{(3-\varepsilon s)(3 - \varepsilon - \varepsilon s)(4-s)}} \Big)(1 + o(1)) \\
&= \Big(\delta_1(\varepsilon) \int_0^1 \frac{ds}{\sqrt{s(s+3/4\,\varepsilon)}} - \varepsilon \int_0^1 \sqrt{\frac{s}{s+3/4\,\varepsilon}} ds \Big) \Big(\frac{1}{2} + O(\delta_2(\varepsilon)) \Big) \\
&= \frac{1}{2} \Big(\delta_1(\varepsilon) \log(1/\varepsilon) + O(\delta_1(\varepsilon)) \Big) \Big(1 + O(\delta_2(\varepsilon)) \Big) \\
&= 2 + O\Big(\log^{-1}(1/\varepsilon) \Big).
\end{aligned}$$

That is, the right hand side of (7.5) equals (-2) times $\cosh(\sqrt{\mu} + O(1))$. The proof is finished.

8 Appendix

8.1 On the rate function I

In this first section of the appendix we revisit the minimization problem (2.1), proving a few technicalities used in the proof of Theorem 2.1.

Proposition 8.1 *Consider the Euler-Lagrange equation corresponding to the minimizer of $I(f;a)$:*

$$\frac{1}{2}(f')^2 = \frac{1}{2} f^4 - f^2 - \alpha f + \frac{1}{2}\beta, \qquad (8.1)$$

recall (2.4). Here α is the multiplier and β a constant of integration. As $a \to \infty$, $\alpha \to 0$ and $\beta \to 1$.

It is natural to suppose that $\alpha \equiv 0$ for all a sufficiently large, but we were unable to prove this.

Proof We first show that $|\beta - 1| = O(1/a)$. Integrating (8.1) and recalling that we are working in the space of mean-zero functions we find

$$\int_{-a}^{a} (df^*/dx)^2 = \int_{-a}^{a} [1 - (f^*)^2]^2 + 2(\beta - 1)a \tag{8.2}$$

for any minimizer $f^* = f^*_a$. It follows that

$$\frac{8}{3} \geq I^*(a) = \int_{-a}^{a} [1 - (f^*)^2]^2 + (\beta - 1)a \geq (\beta - 1)a,$$

which gives us one direction. For the opposite inequality, solve instead for $\int [1 - (f^*)^2]^2$ in (8.2) and substitute that into $I(f^*; a) = I^*(a)$ to yield

$$\frac{8}{3} \geq I^*(a) = \int_{-a}^{a} (df^*/dx)^2 + (1 - \beta)a \geq (1 - \beta)a.$$

Turning to the convergence of α to zero, integrating the preliminary Euler equation, $f'' = 2f^3 - 2f - \alpha$, implies $\alpha = -2\int_{-a}^{a} (f^*)^3$, and so, since f^* and $-f^*$ both minimize, it may from here on be assumed that $\alpha \leq 0$. The important point is to see that $||f^*||_\infty \leq 1$ and that $||f^*||_\infty \to 1$ as $a \to \infty$. First let θ be a maximum of f^*. At any point it is attained (8.1) reads as $0 = (1 - \theta)^2 + 2|\alpha|\theta + (\beta - 1)$ which shows that

$$2\theta|\alpha| \leq 1 - \beta.$$

Now assume that $\theta > 1$. Then there is a point at which $f^* = 1$ and

$$0 \leq 2|\alpha| + \beta - 1 < 2|\alpha|\theta + \beta - 1 \leq 0$$

by (8.1) and the previous display. But this is a contradiction. A similar argument explains why f^* is everywhere greater than -1. Finally denoting $\eta = f^*(x^*)$ where $|f^*(x^*)| = ||f^*||_\infty$ it is plain that $\eta \to 1$ if $I^*(a)$ is to remain bounded. And, again from (8.1) now at x^*, we have that

$$0 = (1 - \eta^2)^2 + 2|\alpha|\eta + (\beta - 1)$$

from which it follows that $\alpha \to 0$.

8.2 Two changes of measure

Here we provide the proof of the result which allowed us to remove the degeneracy, Lemma 3.1, as well as that of the Cameron-Martin formula for P_0^0 invoked in the proof of Proposition 3.1.

As the reader may have observed, Lemma 3.1 is really a type of Rice formula (see [22], [13], or [16] for a more recent account). A proof is provided as we were unable to locate the form of the result needed here. Our proof relies on finite dimensional approximation, and in that direction we first prepare the following.

Lemma 8.1 *Suppose that $X(\cdot)$ is a stationary process, periodic of period one, defined on the lattice $\{k/2^n\}$ and taking values in $\mathbb{Z}/n$. Assume also that X has at least one zero with probability 1. If F is a functional that is invariant under translations, then*

$$E[F(X)] = 2^n E\Big[F(X) N^{-1} \Big| X(0) = 0\Big] P\Big(X(0) = 0\Big)$$

where $N = N(X) =$ the number of zeros of X.

Proof Notice the simple decomposition

$$\begin{aligned} E[F(X), N=1] &= \sum_{k=1}^{2^n} E\Big[F(X), N=1, T_1 = k/2^n\Big] \\ &= 2^n E\Big[F(X), N=1, X(T_1)=0\Big]. \end{aligned}$$

Here T_1 is the (only) zero of X, and we have used the rotation invariance of $F(X)$ in line two. A more general version of this is

$$\begin{aligned} m\, E\Big[F(X), N=m\Big] &= \sum_{i=1}^{m}\sum_{k=1}^{2^n} E\Big[F(X), N=m, T_i = k/2^n\Big] \\ &= \sum_{k=1}^{2^n} E\Big[F(X), N=m, \bigcup_{i=1}^{m}\{T_i = k/2^n\}\Big] \\ &= \sum_{k=1}^{2^n} E\Big[F(X), N=m, X(k/2^n)=0\Big] = 2^n E\Big[F(X), N=m, X(0)=0\Big]. \end{aligned}$$

Now divide both sides of this equality by m and sum to conclude the proof of the lemma.

Proof of Lemma 3.1 If now X is smooth stationary periodic process over the line we define the discrete process Y^n as taking the values $n^{-1}[nX(k/2^n)]$ at $1/2^n, 2/2^n, \ldots$. Applying the previous result to Y^n produces

$$E\Big[F(Y^n)\Big] = 2^n E\Big[F(Y^n) N(Y^n)^{-1}\Big| 0 \le X(0) < 1/n\Big] P\Big(0 \le X(0) < 1/n\Big)$$

Now notice that the number of zeros of Y^n around a zero of X is approximately $2^n/n \cdot |X'(z)|^{-1}$. In fact

$$N(Y^n) \simeq \frac{2^n}{n}\sum_{z\in Z} |X'(z)|^{-1}$$

can be used in the above formula to produce

$$E\Big[F(Y^n)\Big] \simeq E\Big[F(Y^n)\Big(\sum_{z\in Z}|X'(z)|^{-1}\Big)^{-1}\Big| 0 \le X(0) < 1/n\Big]\left(\frac{P(0 \le X(0) < 1/n)}{1/n}\right).$$

This expression has the required limit when $n \uparrow \infty$. The proof is finished.

Finally we have:

Lemma 8.2 *Let $\tilde{E}$ denote the CBM conditioned on both $\int_0^1 p = 0$ and $\int_0^1 \phi p = 0$ for some continuous, mean-zero ϕ. Let also φ be twice continuously differentiable and satisfy $\int_0^1 \varphi = 0$ as well as $\int_0^1 \phi\varphi = 0$. Then*

$$\tilde{E}\left[F(p)\right] = \tilde{E}\left[F(p+\varphi)\exp\left\{\int_0^1 \varphi'' p - \frac{1}{2}\int_0^1 |\varphi'|^2\right\}\right].$$

for all bounded measurable functions F of the path.

Proof The $\tilde{E}$ integral is expressed as an integral with respect to the Brownian bridge measure:

$$\begin{aligned}\tilde{E}\Big[F(p)\Big] &= BM_{00}\left[F\left(p-\int_0^1 p\right) \Big| \int_0^1 p\phi = 0\right] \\ &= \lim_{h\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{B_{h,\varepsilon}}BM_0\left[F\left(p-\int_0^1 p\right), \int_0^1 p\phi\in[0,h],\ p(1)\in[0,\varepsilon]\right]\end{aligned}$$

where $B_{h,\varepsilon} = BM_0[\int_0^1 p\phi \in [0,h], p(1)\in[0,\varepsilon]]$. The usual Cameron-Martin formula may now be applied to the free Brownian integral with the result that

$$\begin{aligned} & BM_0\left[F\left(p-\int_0^1 p\right), \int_0^1 p\phi\in[0,h], p(1)\in[0,\varepsilon]\right] \qquad (8.3) \\ = \ & BM_c\left[F\left(p-\int_0^1 p+\varphi\right)\exp\Big\{-\int_0^1\varphi' dp - \frac{1}{2}\int_0^1|\varphi'|^2\Big\}\int_0^1 p\phi\in[0,h],\ p(1)\in[c,c+\varepsilon]\right],\end{aligned}$$

in which $c=-\varphi(0)$ and use has been made of $\int_0^1\varphi = 0$ and $\int_0^1 \varphi\phi = 0$. Next Itô's formula and the fact that $\int_0^1\varphi''=0$, allows us to write

$$-\int_0^1\varphi'(x)dp(x) = \int_0^1\varphi''(x)\Big(p(x)-\int_0^1 p(x')dx'\Big)dx + \varphi'(0)\Big(p(1)-p(0)\Big).$$

Invoking this move as well as the shift from $p(x)$ under BM_c to $p(x)+c$ under BM_0, the right hand side of (8.3) then reads

$$\begin{aligned} BM_0\Bigg[F\left(p-\int_0^1 p+\varphi\right)\exp\Big\{-\int_0^1\varphi''\Big(p-\int_0^1 p\Big) - \frac{1}{2}\int_0^1|\varphi'|^2\Big\} \\ \exp\Big\{-\phi'(0)p(1)\Big\}, \int_0^1 p\phi\in[0,h],\ p(1)\in[0,0+\varepsilon]\Bigg].\end{aligned}$$

Dividing this object by $B_{h,\varepsilon}$ and performing the limits $\varepsilon\downarrow 0$ and $h\downarrow 0$ completes the proof.

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